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MATHEMATICS

Necessary and Sufficient Conditions for the Boundedness of Genuine Singular Integral Operators in Local Morrey-Type Spaces

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This paper considers the boundedness of Calderon–Zygmund singular integral operators in local and global Morrey-type spaces. We suggest sufficient conditions for the boundedness of Calderon-Zygmund singular integral operators for all admissible parameter values. In the case of local Morrey-type spaces with parameters satisfying certain relations, these sufficient conditions are also necessary for genuine Calderon–Zygmund singular integral operators.

Definition 1 [2]. Let $0 < p, \theta \le \infty$, and let w be a nonnegative measurable function on $(0, \infty)$. The local and global Morrey-type spaces $LM_{p\theta, w}$ and $GM_{p\theta, w}$ are defined as the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p\theta, w}} \equiv \|f\|_{LM_{p\theta, w}(\mathbb{R}^n)} = \big\| w(r) \|f\|_{L_p(B(0, r))} \big\|_{L\theta(0, \infty)}$$

and

$$\|f\|_{GM_{p\theta, w}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p\theta, w}},$$

respectively.

Definition 2 [2]. For $0 < p, \theta \le \infty$, Ω_θ d*w*enotes the set of all nonnegative measurable functions w on $(0, \infty)$ which are not equivalent to zero and satisfy the condition $\|w(r)\|_{L_\theta(t, \infty)} < \infty$ for some $t > 0$. Let $\Omega_{p\theta}$ denote the set of all nonnegative measurable functions w on $(0, \infty)$ not equivalent to zero and such that, for some $t_1, t_2 > 0$,

$$\|w(r)\|_{L_\theta(t_1, \infty)} < \infty \quad \text{and} \quad \|w(r) r^{n/p}\|_{L_\theta(0, t_2)} < \infty.$$

In the case of local Morrey-type spaces, we assume that $w \in \Omega_\theta$, and in the case of global Morrey-type spaces, we assume that $w \in \Omega_{p\theta}$, because if these assumptions do not hold, then the spaces $LM_{p\theta, w}$ and $GM_{p\theta, w}$ respectively, are trivial (contain only functions equivalent to zero); see [3].

Note that if $w(r) = 1$, then $LM_{p\infty, 1} = GM_{p\infty, 1} = L_p(\mathbb{R}^n)$.

The spaces $GM_{p\infty, r^{-\lambda/p}}$ with $0 < \lambda < n$ are the classical Morrey spaces, which were first considered in [10]; they were applied to study the local behavior of solutions to second-order elliptic differential equations.

Let T be a Calderon–Zygmund operator, i.e., a bounded linear operator from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ which takes each infinitely differentiable compactly supported function f to a function $Tf \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \text{ almost everywhere in } \operatorname{supp} f.$$

Here, $K(x, y)$ is a function continuous outside the diagonal and such that, for some $c_1 > 0$ and $0 < \varepsilon \le 1$,

$$|K(x, y)| \le c_1 |x - y|^{-n}$$

for any different $x, y \in \mathbb{R}^n$, $x \ne y$ and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le c_1 \left(\frac{|x - x'|}{|x - y|}\right)^{\varepsilon} |x - y|^{-n},$$

if $2|x - x'| \le |x - y|$. These operators were introduced in [1].

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We say that T is a genuine Calderon–Zygmund operator if it is a Calderon–Zygmund operator and, for $n \geq 2$, there exist $c_2, c_3 > 0$ and a rotation $\mathcal{R}$ such that

$$K(x, y) \geq \frac{c_2}{|x-y|^n}$$

for any $x \in \mathbb{R}^n$ and any $y \in C_x = x + \mathcal{R}(C)$, where

$$C = \{y = (y_1, y_2, \dots, y_n) \equiv (\bar{y}, y_n) \in \mathbb{R}^n\colon y_n > c_3|\bar{y}|\}.$$

If $n = 1$, then we assume the existence of a $c_2 > 0$ such that

$$K(x, y) \geq \frac{c_2}{|x-y|}$$

for any $x \in \mathbb{R}$ and any $y > x$ or for any $x \in \mathbb{R}$ and any $y < x$.

Examples of genuine Calderon–Zygmund operators are the Hilbert operator with $K(x, y) = \frac{1}{x-y}$ (for $n \geq 2$) and the Calderon–Zygmund operator with kernel

$$K(x, y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n},$$

where Ω is a continuous function on the unit sphere homogeneous of degree zero and such that $\Omega \not\equiv 0$ and $\int\limits_{S^{n-1}} \Omega(\eta)d\eta = 0$ (for $n \geq 2$).

Mizuhara [9], Nakai [11], and Guliev [7] (see also [8]) obtained sufficient conditions for the boundedness of Calderon–Zygmund operators as operators from $GM_{p\infty, w_1}$ to $GM_{p\infty, w_2}$.

Naturally, it is of interest to find necessary and sufficient conditions on w_1 and w_2 ensuring the boundedness of the operator T from a Morrey-type space to another such space. For this purpose, we reduce the problem of the boundedness of T for Morrey-type spaces to the boundedness of the Hardy operator on weighted L_p-spaces on the cone of nonnegative nonincreasing functions. A similar approach was used in [2–8] for obtaining necessary and sufficient conditions for the boundedness of maximal operators, fractional maximal operators, and Riesz potentials from a Morrey-type space to another such space. As in [7, 8], we use the following inequality.

Lemma 1. *If* $1 < p < \infty$ *and* $\gamma \geq 1$, *then, for all* $r > 0$ *and all* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$,

$$\|Tf\|_{L_p(B(0,r))} = c_4 r^{n/p} \int\limits_{\gamma r}^{\infty} r^{-n/p-1} \|f\|_{L_p(B(0,t))} dt,$$

where $c_4 > 0$ *does not depend on r and f.*

Corollary. *If* $1 < p < \infty$, $0 < \delta < \frac{n}{p}$, *and* $\gamma \geq 1$, *then, for all* $r > 0$ *and all* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$,

$$\|Tf\|_{L_p(B(0,r))} \leq c_5 r^{n/p-\delta} \left(\int\limits_{\gamma r}^{\infty} \left(\int\limits_{B(0,t)} |f(x)|^p dx \right) \frac{dt}{t^{n-\delta p+1}} \right)^{1/p},$$

where $c_5 > 0$ *does not depend on r and f.*

Let H denote the Hardy operator

$$(Hg)(r) = \int\limits_0^r g(t)dt \quad \text{where} \quad 0 < r < \infty,$$

and let $L_{p,v}(0, \infty)$ (where v is a nonnegative measurable weight function) be the space of all measurable functions f on $(0, \infty)$ for which $\|f\|_{L_{p,v}(0,\infty)} = \|vf\|_{L_p(0,\infty)} < \infty$.

Lemma 2. *For any* $1 < p < \infty$, $0 < \theta \leq \infty$, *and* $w \in \Omega_\theta$, *there exists a* $c_6 > 0$ *such that, for all* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$,

$$\|Tf\|_{LM_{p\theta,w}} \leq c_6 \|H\hat{g}\|_{L_{\theta,\hat{v}}(0,\infty)},$$

where

$$\hat{g}(t) = \|f\|_{L_p\left(B\left(0, t^{-\frac{p}{n}}\right)\right)}$$

and

$$\hat{v}(r) = w\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}\left(\frac{n}{p}+\frac{1}{\theta}\right)-\frac{1}{\theta}}.$$

Lemma 3. *For any* $1 < p < \infty$, $0 < \theta \leq \infty$, $0 < \delta < \frac{n}{p}$, *and* $w \in \Omega_\theta$, *there exists a* $c_7 > 0$ *such that, for all* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$,

$$\|Tf\|_{LM_{p\theta,w}} \leq c_7 \|Hg_\delta\|^{1/p}_{L_{\frac{\theta}{p}, v_\delta}(0,\infty)},$$

where

$$g_\delta = \int\limits_{B(0, t^{1/(\delta p - n)})} |f(y)|^p dy,$$

and

$$v_\delta(r) = \left[w\left(r^{\frac{1}{\delta p - n}}\right) r^{\frac{1}{\delta p - n}\left(\frac{n}{p} - \delta + \frac{1}{\theta}\right) - \frac{1}{\theta}} \right]^n.$$

Theorem 1. *Suppose that* $1 < p < \infty$, $0 < \theta_1, \theta_2 \leq \infty$, $0 < \delta < \frac{n}{p}$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$. *Suppose also that*

$$v_{1,\delta}(r) = \left[w_1\left(r^{\frac{1}{\delta p-n}}\right)r^{\frac{1}{(\delta p-n)\theta_1}-\frac{1}{\theta_1}}\right]^p,$$

$$v_{2,\delta}(r) = \left[w_2\left(r^{\frac{1}{\delta p-n}}\right)r^{\frac{1}{\delta p-n}\left(\frac{n}{p}-\delta+\frac{1}{\theta_2}\right)-\frac{1}{\theta_2}}\right]^p$$

and

$$\hat{v}_1(r) = w_1\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n\theta_1}-\frac{1}{\theta_1}},$$

$$\hat{v}_2(r) = w_2\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}\left(\frac{n}{p}+\frac{1}{\theta_2}\right)-\frac{1}{\theta_2}}.$$

Finally, suppose that the operator H is bounded as an operator from $L_{\frac{\theta_1}{p}, v_{1,\delta}}(0, \infty)$ *to* $L_{\frac{\theta_2}{p}, v_{2,\delta}}(0, \infty)$ *or from* $L_{\theta_1, \hat{v}_1}(0, \infty)$ *to* $L_{\theta_2, \hat{v}_2}(0, \infty)$ *on the cone of all nonincreasing nonnegative functions* φ *on* $(0, \infty)$ *satisfying the condition* $\lim\limits_{t \to \infty} \varphi(t) = 0$.

Then, the operator T is bounded as an operator from $LM_{p\theta_1, w_1}$ *to* $LM_{p\theta_2, w_2}$ *and from* $GM_{p\theta_1, w_1}$ *to* $GM_{p\theta_2, w_2}$ (*in the latter case*, it is assumed that $w_1 \in \Omega_{p, \theta_1}$ *and* $w_2 \in \Omega_{p, \theta_2}$).

For most parameter values, necessary and sufficient conditions on the weights function v_1 and v_2 which ensure that

$$\|H\varphi\|_{L_{\frac{\theta_2}{p}, v_2}(0, \theta)} \le c_8 \|\varphi\|_{L_{\frac{\theta_1}{p}, v_1}(0, \theta)} \qquad (1)$$

for any nonnegative nonincreasing functions φ, where $c_8 > 0$ does not depend on φ, are known (see, e.g., [12]). The application of these conditions gives sufficient conditions for the boundedness of the operator T from $LM_{p\theta_1, w_1}$ to $LM_{p\theta_2, w_2}$ and from $GM_{p\theta_1, w_1}$ to $GM_{p\theta_2, w_2}$. One of such conditions is as follows.

Theorem 2. *Suppose that* $1 < p < \infty$, $0 < \theta_2 \le \infty$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_{\theta_2}$, *and, for some* $c_9 > 0$,

$$\left\|w_1^{-1}(t)r^{-\frac{n}{p}-1}\right\|_{L_1(r, \infty)} \le c_9 w_1^{-1}(r) r^{-\frac{n}{p}}$$

for any $r > 0$ *and*

$$\left\|w_2(r)r^{\delta+\frac{n}{p}}\left\|w_1^{-1}(t)t^{-\delta-\frac{n+1}{p}}\right\|_{L_p(r, \infty)}\right\|_{L_{\theta_2}(0, \infty)} < \infty$$

for any $\delta > 0$ (*if* $\theta_2 < \infty$) *or for* $\delta = 0$ (*if* $\theta_2 = \infty$).

Then, the operator T is bounded from $LM_{p\infty, w_1}$ *to* $LM_{p\theta_2, w_2}$ *and from* $GM_{p\infty, w_1}$ *to* $GM_{p\theta_2, w_2}$ (*in the latter case, it is assumed that* $w_1 \in \Omega_{p, \infty}$ *and* $w_2 \in \Omega_{p\theta_2}$.)

For $\theta_2 = \infty$, a somewhat stronger result was obtained in [7, 8] by Guliev, who proved that if there exists a $c_{10} > 0$ such that

$$\left\|w_1^{-1}(r)r^{-n/p-1}\right\|_{L_1(t, \infty)} \le c_{10} w_2^{-1}(t)t^{-n/p},$$

for any $t > 0$, then the operator T is bounded from $LM_{p\infty, w_1}$ to $LM_{p\infty, w_2}$ and from $GM_{p\infty, w_1}$ to $GM_{p\infty, w_2}$. This result covers earlier results of Mizuhara [9] and Nakai [11] for the case of $w_1 = w_2 = w$, where w satisfies the doubling condition.

Necessary and sufficient conditions on v_1 and v_2 ensuring (1) do not directly imply necessary and sufficient conditions for the boundedness of the operator T from $LM_{p\theta_1, w_1}$ to $LM_{p\theta_2, w_2}$. However, for some parameter values (namely, for $1 < p < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, and $\theta_1 \le 1$), such conditions can be obtained. Note that, in this case, necessary conditions (which coincide with sufficient ones) for inequality (1) to hold for nonnegative nonincreasing functions are obtained by taking $\varphi = \chi_{(0, t)}$ with any $t > 0$, where $\chi_{(0, t)}$ is the characteristic function of the interval $(0, t)$.

Theorem 3. *If* $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$, *then the following assertions are valid.*

(i) The condition

$$\left\|w_2(r)\left(\frac{r}{t+r}\right)^{n/p}\right\|_{L_{\theta_2}(0, \infty)} \le c_{11}\|w_1\|_{L_{\theta_1}(t, \infty)} \qquad (2)$$

for all $t > 0$, *where* $c_{11} > 0$ *does not depend on t, is necessary for the boundedness of a genuine Calderon–Zygmund operator T from* $LM_{p\theta_1, w_1}$ *to* $LM_{p\theta_2, w_2}$.

(ii) *If* $\theta_1 \le \theta_2$ *and* $\theta_1 \le 1$, *then condition* (2) *is sufficient for the boundedness of a Calderon–Zygmund operator T from* $LM_{p\theta_1, w_1}$ *to* $LM_{p\theta_2, w_2}$ *and from* $GM_{p\theta_1, w_1}$ *to* $GM_{p\theta_2, w_2}$ (*in the latter case, it is assumed that* $w_1 \in \Omega_{p\theta_1}$ *and* $w_2 \in \Omega_{p\theta_2}$).

(iii) *In particular, if T is a genuine Calderon–Zygmund operator,* $\theta_1 \le \theta_2$, *and* $\theta_1 \le 1$, *then condition* (2) *is necessary and sufficient for the boundedness of T from* LM_{θ_1, w_1} *to* LM_{θ_2, w_2}.

Theorem 4. *If T is a Calderon–Zygmund operator,* $1 < p < \infty$, $0 < \theta \le \infty$, *and* $w \in \Omega_\theta$, *then the condition*

$$w \in L_\theta(0, \theta) \qquad (3)$$

i*s sufficient for the boundedness of the operator T from* $L_p(\mathbb{R}^n)$ *to* $LM_{p\theta, w}$ *and from* $L_p(\mathbb{R}^n)$ *to* $GM_{p\theta, w}$; *for a genuine Calderon–Zygmund operator, it is also necessary.* (*In the case of the spaces* $GM_{p\theta, w}$ *it is assumed that* $w \in \Omega_{p\theta}$.)

Theorems 3 and 4 are not valid for $p = 1$; however, in this case, they have analogues, which can be obtained by considering larger weak Morrey-type spaces.

For $0 < p < \infty$ and a measurable set $G \subset \mathbb{R}^n$, let $WL_p(G)$ denote the weak space $L_p(G)$ of all measurable functions f on G for which

$$\|f\|_{WL_p(G)} = \sup_{t>0} t(\mathrm{mes}\{x \in G\colon |f(x) > t|\})^{1/p} < \infty.$$

If $p = \infty$, then $WL_\infty(G) = L_\infty(G)$.

Definition 3 [2]. *Let* $0 < p, \theta \le \infty$, *and let* w *be a nonnegative measurable function on* $(0, \infty)$. By $LWM_{p\theta,w}$ *and* $GWM_{p\theta,w}$ *e denote the local and global weak Morrey-type spaces of all functions* $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ *with finite quasinorms*

$$\|f\|_{LWM_{p\theta,w}} = \|f\|_{LWM_{p\theta,w}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{WL_p(B(0,r))}\big\|_{L_\theta(0,\infty)}$$

and

$$\|f\|_{GWM_{p\theta,w}} = \sup_{x \in \mathbb{R}^n} \|f(x+\cdot)\|_{LWM_{p\theta,w}}.$$

respectively.

Theorem 5. *Suppose that* $1 \le p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$. *Then, the following assertions hold.*

(i) *Condition* (2) *is necessary for the boundedness of any genuine Calderon–Zygmund operator T from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$.

(ii) *If* $\theta_1 \le \theta_2$ *and* $\theta_1 \le 1$, *then condition* (2) *is sufficient for the boundedness of any Calderon–Zygmund operator T from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GWM_{p\theta_2,w_2}$ (*in the latter case, it is assumed that* $w_1 \in \Omega_{p\theta_1}$ *and* $w_2 \in \Omega_{p\theta_2}$).

(iii) *In particular, if T is a genuine Calderon–Zygmund operator,* $\theta_1 \le \theta_2$, *and* $\theta_1 \le 1$, *then condition* (2) *is necessary and sufficient for the boundedness of the operator T from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$.

Theorem 6. *If T is a* (genuine) *Calderon–Zygmund operator,* $1 \le p < \infty$, $0 < \theta \le \infty$, *and* $w \in \Omega_\theta$, *then condition* (3) *is sufficient* (*respectively, necessary and sufficient*) *for the boundedness of the operator T from* $L_p(\mathbb{R}^n)$ *to* $LWM_{p\theta,w}$ *and from* $L_p(\mathbb{R}^n)$ to $GWM_{p\theta,w}$. (*In the latter case, it is assumed that* $GWM_{p\theta,w}$).

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