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POINTWISE AND INTEGRAL ESTIMATES FOR THE *B*-RIESZ POTENTIAL IN TERMS OF *B*-MAXIMAL AND *B*-FRACTIONAL MAXIMAL FUNCTIONS

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Abstract: We study the maximal and fractional maximal functions and Riesz potentials that are generated by the generalized shift operator associated with the Laplace–Bessel operator. We obtain some pointwise and integral estimates that give a relation between the B-maximal and B-fractional maximal functions and B-Riesz potentials and extend the available results to the objects of a more general nature. Basing on these results, we prove interpolation theorems for the B-fractional maximal functions and B-Riesz potentials.

Keywords: generalized shift operator, B-maximal function, B-fractional maximal function, B-Riesz potential, Sobolev theorem

This article addresses some problems of harmonic analysis that are associated with the Laplace–Bessel operator Δ_B. Unlike the classical case of the convolution-like operators generated by a usual shift τ^h, $h \in \mathbb{R}^n$ $(\tau^h\varphi(x) = \varphi(x-h))$, we consider convolution structures generated by a generalized shift adjusted to the Laplace–Bessel operator. The study of the Laplace–Bessel operator requires the use of some classes of special functions and the corresponding Fourier–Bessel integral transformations. The use of the Fourier–Bessel transformation for Δ_B makes possible to get a fundamental solution as a B-Riesz potential (see [1–3]).

Here we deal with B-maximal and B-fractional maximal functions and B-Riesz potentials. We prove the $(L_{p,\gamma}, L_{q,\gamma})$-boundedness of B-fractional maximal functions, obtain some pointwise and integral estimates establishing relations between B-maximal and B-fractional maximal functions and B-Riesz potentials, and prove Sobolev's theorem in the limit case $\big(p = Q/\alpha\big)$. Basing on these, we prove interpolation theorems for B-fractional maximal functions and B-Riesz potentials.

1. Definitions and Preliminaries

Let $\mathbb{R}^N$ be the N-dimensional Euclidean space of points $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $1 \le n \le N$, $N \ge 2$; let $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x'' = (x_{n+1}, \dots, x_N) \in \mathbb{R}^{N-n}$, $x = (x', x'') \in \mathbb{R}^N$; $\mathbb{R}^N_{n,+} = \{x = (x', x'') \in \mathbb{R}^N : x_1 > 0, \dots, x_n > 0\}$; $B(x,r) = \big\{y \in \mathbb{R}^N_{n,+} : |x-y| < r\big\}$; $\gamma = (\gamma_1, \dots, \gamma_n)$, $|\gamma| = \gamma_1 + \dots + \gamma_n$, $\gamma_1 > 0, \dots, \gamma_n > 0$; and $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_n^{\gamma_n}$.

In the case $n = N$, we assume that $x = x'' \in \mathbb{R}^N_+$, $\mathbb{R}^N_+ \equiv \mathbb{R}^N_{n,+} = \{x \in \mathbb{R}^N : x_1 > 0, \dots, x_N > 0\}$, $\gamma = (\gamma_1, \dots, \gamma_N)$.

Given a measurable set $E \subset \mathbb{R}^N_{n,+}$, put $|E|_\gamma = \int_E (x')^\gamma dx$. Then

$$|B(0,r)|_\gamma = \omega(N,n,\gamma) r^Q, \quad Q = N + |\gamma|,$$

where

$$\omega(N,n,\gamma) = \int\limits_{B(0,1)} (x')^\gamma dx = \frac{\pi^{\frac{N-n}{2}}}{2^n} \prod_{i=1}^n \frac{\Gamma\big(\frac{\gamma_i+1}{2}\big)}{\Gamma\big(\frac{\gamma_i}{2}\big)}.$$

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The *generalized shift operator* T^y is defined as

$$T^y f(x) = C_{\gamma,n} \int_0^\pi \cdots \int_0^\pi f((x_1,y_1)_{\alpha_1},\dots,(x_n,y_n)_{\alpha_n},x''-y'')\,d\nu(\alpha),$$

where

$$C_{\gamma,n} = \pi^{-\frac{n}{2}} \prod_{i=1}^n \Gamma((\gamma_i+1)/2)\Gamma^{-1}(\gamma_i/2),$$

$$(x_i,y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2}, \quad 1 \le i \le n,$$

$$d\nu(\alpha) = \prod_{i=1}^n \sin^{\gamma_i - 1}\alpha_i\,d\alpha_1 \dots d\alpha_n, \quad 1 \le n \le N.$$

Note that T^y is closely related to the Laplace–Bessel operator Δ_B (see [2]). Kipriyanov and Ivanov [2] showed that the volume potential

$$u(x) = \int_{R^N_{n,+}} |y|^{2-Q} T^y f(x)(y')^\gamma\,dy$$

is a solution to the B-elliptic equation

$$\Delta_B u(x) = f(x),$$

where

$$\Delta_B = \sum_{i=1}^n B_i + \sum_{i=n+1}^N \frac{\partial^2}{\partial x_i^2}, \quad B = (B_1,\dots,B_n),$$

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_i > 0,\ i = 1,\dots,n.$$

As we see, the solution to this problem contains a transformation operator that was introduced by Levitan [4] in the one-dimensional case and called the *generalized shift operator*. The extensive use of the method of transformation operators seems to stem from [4–6] etc. A number of important results in this direction were established by Kipriyanov and his students for B-elliptic, B-parabolic, and B-hyperbolic equations (see [3] for details). The fact that a solution to the problem was given as the volume potential substantiates the necessity of studying various properties of the potentials that are solutions to some singular differential equations.

Denote by $L_{p,\gamma} = L_{p,\gamma}(\mathbb{R}^N_{n,+})$ the space of measurable functions $f(x)$, $x \in \mathbb{R}^N_{n,+}$, with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^N_{n,+}} |f(x)|^p (x')^\gamma\,dx \right)^{1/p}, \quad 1 \le p < \infty.$$

Put $L_{\infty,\gamma}(\mathbb{R}^N_{n,+}) = L_\infty(\mathbb{R}^N_{n,+})$, where $L_\infty(\mathbb{R}^N_{n,+})$ is the class of all essentially bounded functions f with norm $\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^N_{n,+}} |f(x)|$.

We now define the space $BMO_\gamma(\mathbb{R}^N_{n,+})$ (see [7, 8]). Given $f \in L^{\rm loc}_{1,\gamma}(\mathbb{R}^N_{n,+})$, put

$$f_{B(0,r)}(x) = |B(0,r)|_\gamma^{-1} \int_{B(0,r)} T^y f(x)(y')^\gamma\,dy.$$

Here $B(0,r) = \{ y \in \mathbb{R}^N_{n,+} : |y| < r \}$.

We say that $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^N_{n,+})$ *belongs to* $BMO_\gamma(\mathbb{R}^N_{n,+})$ if

$$\|f\|_{*,\gamma} = \sup_{x,r} |B(0,r)|^{-1}_\gamma \int\limits_{B(0,r)} |T^y f(x) - f_{B(0,r)}(x)|(y')^\gamma \, dy < \infty.$$

It is natural to define the convolution (B-convolution) generated by the generalized shift operator. If φ and ψ are integrable functions on $\mathbb{R}^N_{n,+}$ then

$$(\varphi \otimes \psi)(x) = \int\limits_{\mathbb{R}^N_{n,+}} \varphi(y) T^y \psi(x)(y')^\gamma \, dy.$$

Considering the properties of generalized shifts, it is easy to show $\varphi \otimes \psi = \psi \otimes \varphi$ and the Young inequality for a B-convolution:

$$\|f \otimes g\|_{r,\gamma} \le \|f\|_{p,\gamma} \|g\|_{q,\gamma}, \quad 1 \le p, q, r \le \infty, \quad 1/p + 1/q = 1/r + 1,$$

and, for $1 \le p \le \infty$, $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$, $y \in \mathbb{R}^N_{n,+}$,

$$\|T^y f(\cdot)\|_{p,\gamma} \le \|f\|_{p,\gamma} \tag{1}$$

(for example, see [9]).

Let $f : \mathbb{R}^N_{n,+} \to \mathbb{R}$ be a measurable function. Then the nondecreasing γ-rearrangement of f is defined as follows:

$$f^*_\gamma(t) = \inf\{s > 0 : f_{*,\gamma}(s) \le t\} \quad \forall t \in [0, \infty),$$

where $f_{*,\gamma}$ is the γ-distribution of f,

$$f_{*,\gamma}(t) = |\{x \in \mathbb{R}^N_{n,+} : |f(x)| > t\}|_\gamma \quad \forall t \in [0, \infty).$$

For the γ-rearrangement of f we have (see [10–12] for details):

(1) if $0 < p < \infty$ then

$$\int\limits_{\mathbb{R}^N_{n,+}} |f(x)|^p (x')^\gamma \, dx = \int\limits_0^\infty (f^*_\gamma(t))^p \, dt;$$

(2) for every $t > 0$,

$$\sup_{|E|_\gamma = t} \int\limits_E |f(x)|(x')^\gamma \, dx = \int\limits_0^t f^*_\gamma(s) \, ds;$$

(3)

$$\int\limits_{\mathbb{R}^N_{n,+}} |f(x) g(x)|(x')^\gamma \, dx \le \int\limits_0^\infty f^*_\gamma(t) g^*_\gamma(t) \, dt.$$

The weak $L_{p,\gamma}$-space $WL_{p,\gamma}(\mathbb{R}^N_{n,+})$, $1 \le p < \infty$, is defined as the set of locally summable functions $f(x)$, $x \in \mathbb{R}^N_{n,+}$, with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r f_{*,\gamma}(r)^{1/p}.$$

Define the function

$$f^{**}_\gamma(t) = \frac{1}{t} \int\limits_0^t f^*_\gamma(s) \, ds, \quad t > 0,$$

on $(0, \infty)$. In regard to $f^{**}_\gamma(t)$, we have the following inequality (see [13]):

$$(f + g)^{**}_\gamma(t) \le f^{**}_\gamma(t) + g^{**}_\gamma(t).$$

Below we will need a few lemmas for the proof of Sobolev's theorem in the limit case $p = Q/\alpha$.

Lemma 1 [11, 14]. *Let f and g be positive measurable functions on $\mathbb{R}^N_{n,+}$. Then*

$$(f \otimes g)^{**}_\gamma(t) \le C_{k,\gamma}\bigg(f^{**}_\gamma(t)\int\limits_0^t g^{**}_\gamma(u)\,du + \int\limits_t^\infty f^*_\gamma(u)g^{**}_\gamma(u)\,du\bigg) \tag{2}$$

for all $t > 0$.

Lemma 2 [11, 14]. *Let $0 < \alpha < Q$ and $K(x) = |x|^{\alpha-Q}$, $x \in \mathbb{R}^N_{n,+}$. Then*

$$K^*_\gamma(t) = \left(\frac{\omega(N,n,\gamma)}{t}\right)^{\frac{Q-\alpha}{Q}} \quad \text{and} \quad K^{**}_\gamma(t) = \frac{Q}{\alpha}K^*_\gamma(t).$$

Lemma 3 [13]. *Let $a(s,t)$ be a nonnegative measurable function on $(-\infty,+\infty)\times[0,+\infty)$ such that*

$$a(s,t) \le 1, \tag{3}$$

$$\operatorname*{ess\,sup}_{t>0}\bigg(\int\limits_{-\infty}^0 + \int\limits_t^\infty a(s,t)^{p'}\,ds\bigg)^{1/p'} = b < \infty \tag{4}$$

for all $0 < s < t$ almost everywhere. Then there is a constant $C_0 = C_0(p,b)$ such that, for $\phi \ge 0$ and

$$\int\limits_{-\infty}^\infty \phi(s)^p\,ds \le 1, \tag{5}$$

we have

$$\int\limits_0^\infty e^{-F(t)}dt \le C_0, \tag{6}$$

where

$$F(t) = t - \bigg(\int\limits_{-\infty}^\infty a(s,t)\phi(s)\,ds\bigg)^{p'}. \tag{7}$$

2. $(L_{p,\gamma}, L_{q,\gamma})$-Boundedness of B-Fractional Maximal Functions

In this section, we will address the boundedness of the B-maximal functions $M_\gamma f(x)$ and B-fractional maximal functions $M^\alpha_\gamma f(x)$ generated by a generalized shift in $L_{p,\gamma}$. The B-maximal function is defined as (see [7]):

$$M_\gamma f(x) = \sup_{r>0}|B(0,r)|^{-1}_\gamma \int\limits_{B(0,r)} T^y|f(x)|(y')^\gamma\,dy.$$

Also consider the B-fractional maximal function

$$M^\alpha_\gamma f(x) = \sup_{r>0}|B(0,r)|^{\frac{\alpha}{Q}-1}_\gamma \int\limits_{B(0,r)} T^y|f(x)|(y')^\gamma\,dy, \quad 0 \le \alpha < Q.$$

Note that $M^0_\gamma f = M_\gamma f$ for $\alpha = 0$.

We have

Theorem 1. *Let $0 \le \alpha < Q$, $1 \le p \le Q/\alpha$, and $1/p - 1/q = \alpha/Q$.*

1. If $p = 1$ and $f \in L_{1,\gamma}(\mathbb{R}^N_{n,+})$ then

$$\int\limits_{\{x\in\mathbb{R}^N_{n,+}:M^\alpha_\gamma f(x)>\tau\}} (x')^\gamma\,dx \le \left(\frac{C_1}{\tau}\int\limits_{R^N_{n,+}} |f(x)|(x')^\gamma\,dx\right)^q \tag{8}$$

for all $\tau > 0$, where C_1 is a constant independent of f.

2. If $1 < p < \frac{Q}{\alpha}$ and $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ then $M^\alpha_\gamma f \in L_{q,\gamma}(\mathbb{R}^N_{n,+})$ and

$$\left(\int\limits_{\mathbb{R}^N_{n,+}} \big(M^\alpha_\gamma f(x)\big)^q (x')^\gamma\,dx\right)^{1/q} \le C_2\left(\int\limits_{\mathbb{R}^N_{n,+}} |f(x)|^p (x')^\gamma dx\right)^{1/p}, \tag{9}$$

where C_2 is a constant independent of f.

3. If $p = \frac{Q}{\alpha}$ and $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ then $M^\alpha_\gamma f \in L_{\infty,\gamma}(\mathbb{R}^N_{n,+})$ and

$$\sup_{x\in\mathbb{R}^N_{n,+}} M^\alpha_\gamma f(x) \le \left(\int\limits_{\mathbb{R}^N_{n,+}} |f(x)|^p (x')^\gamma\,dx\right)^{1/p}. \tag{10}$$

Proof. Consider the case when $p = \frac{Q}{\alpha}$. Using Hölder's inequality and (1) for $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$, we see that

$$|B(0,r)|_\gamma^{\alpha/Q-1} \int\limits_{B(0,r)} T^y|f(x)|(y')^\gamma\,dy$$

$$\le |B(0,r)|_\gamma^{\frac{\alpha}{Q}-1+1-\frac{1}{p}}\left(\int\limits_{B(0,r)} (T^y|f(x)|)^p (y')^\gamma\,dy\right)^{1/p} \le \|T^y f\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}.$$

The claim 3 of Theorem 1 is thus immediate.

Proceed to the proof of claims 2 and 3 of the theorem. Introduce the fractional maximal function on a space of homogeneous type. By a space of homogeneous type we mean a topological space X with some continuous pseudometric ρ and some positive measure μ that satisfies the doubling condition

$$\mu(B(x,2r)) \le c\mu(B(x,r)), \tag{11}$$

where c is independent of x and $r > 0$. Here $B(x,r) = \{y \in X : \rho(x,y) < r\}$. Let (X,ρ,μ) be a space of homogeneous type. Put

$$M^\beta_\mu f(x) = \sup_{r>0} \mu(B(x,r))^{\beta-1} \int\limits_{B(x,r)} |f(y)|d\mu(y), \quad 0 \le \beta < 1.$$

It is known that the fractional maximal operator M^β_μ, $0 \le \beta < 1$, is an operator of weak type $(1,q)$, $1 - 1/q = \beta$; i.e.,

$$\mu\big\{x \in X : M^\beta_\mu f(x) > \tau\big\} \le \left(\frac{C'_1}{\tau}\int\limits_X |f(x)|\,d\mu(x)\right)^q, \quad 1 - \frac{1}{q} = \beta; \tag{12}$$

and of strong type (p,q) for $1 < p < 1/\beta$, $1/p - 1/q = \beta$ (see [15, 16]); i.e.,

$$\left(\int\limits_X |M^\beta_\mu f(x)|^q\,d\mu(x)\right)^{1/q} \le C'_2\left(\int\limits_X |f(x)|^p\,d\mu(x)\right)^{1/p}, \quad \frac{1}{p} - \frac{1}{q} = \beta. \tag{13}$$

To prove Theorem 1, we will use these statements. In our case $X = \mathbb{R}^N_{n,+}$, $\rho(x,y) = |x-y|$, $\beta = \frac{\alpha}{Q}$, $0 \le \alpha < Q$, and $d\mu(x) = (x')^\gamma\,dx$. It is clear that this measure satisfies the doubling condition (11).

Let us show that

$$M^\alpha_\gamma f(x) \le c_1 M^\beta_\mu f(x),$$

where $c_1 = \omega(N,n,\gamma)^{\alpha/Q-1} 2^{Q-\alpha}(1+c_2)$, $c_2 = \frac{C_{\gamma,1}}{\gamma} 2^{(\frac{\gamma}{2}-1)_+ +1}$, $a_+ = \max\{a,0\}$.

Put

$$M^\alpha_{\gamma,r} f(x) = |B(0,r)|_\gamma^{\frac{\alpha}{Q}-1} \int\limits_{B(0,r)} T^y |f(x)| (y')^\gamma\,dy.$$

We have

$$M^\alpha_{\gamma,r} f(x) = |B(0,r)|_\gamma^{\frac{\alpha}{Q}-1} \int\limits_{\mathbb{R}^N_{n,+}} T^y |f(x)| \chi_{B(0,r)}(y) (y')^\gamma\,dy$$

$$= |B(0,r)|_\gamma^{\frac{\alpha}{Q}-1} \int\limits_{\mathbb{R}^N_{n,+}} |f(y)| T^y \chi_{B(0,r)}(x) (y')^\gamma\,dy,$$

where

$$T^y \chi_{B(0,r)}(x) = c_\gamma \int\limits_0^\pi \cdots \int\limits_0^\pi \chi_{B(0,r)}((x_1,y_1)_{\alpha_1}, \ldots, (x_n,y_n)_{\alpha_n}, x''-y'')\,d\nu(\alpha),$$

χ_A is the characteristic function of $A \subset \mathbb{R}^N_{n,+}$.

Note that $T^y \chi_{B(0,r)}(x) = 0$ for every $y \in \mathbb{R}^N_{n,+} \setminus B(x,r)$, i.e., the support of $T^y \chi_{B(0,r)}(x)$ lies in the ball $B(x,r)$.

Considering the properties of B-convolutions and the fact that the support of $T^y \chi_{B(0,r)}(x)$ lies in $B(x,r)$, we have

$$M^\alpha_{\gamma,r} f(x) = |B(0,r)|_\gamma^{\frac{\alpha}{Q}-1} \int\limits_{B(x,r)} |f(y)| T^y \chi_{B(0,r)}(x) (y')^\gamma\,dy.$$

In what follows, we will assume for simplicity that $n=1$. In this case $\mathbb{R}^N_{n,+} = \mathbb{R}^N_+$. Show that [8] for all $x \in \mathbb{R}^N_+$, $r > 0$ and for all $y \in B(x,r)$,

$$0 \le T^y \chi_{B(0,r)}(x) \le \min\{1, c_2 r^\gamma / x_1^\gamma\}. \tag{14}$$

Note the obvious inequality

$$0 \le T^y \chi_{B(0,r)}(x) \le 1 \tag{15}$$

for $x \in \mathbb{R}^N_{n,+}$ and $y \in B(x,r)$.

We have

$$T^y \chi_{B(0,r)}(x) \le C_{\gamma,1} \int\limits_{\{\alpha\in(0,\pi):(x_1,y_1)^2_\alpha + |x''-y''|^2 < r^2\}} \sin^{\gamma-1}\alpha\,d\alpha$$

$$\le C_{\gamma,1} \int\limits_{\{\alpha\in(0,\pi):(x_1,y_1)_\alpha < r\}} \sin^{\gamma-1}\alpha\,d\alpha = C_{\gamma,1} \int\limits_{\left\{\alpha\in(0,\pi):\frac{x_1^2+y_1^2-r^2}{2x_1y_1} < \cos\alpha\right\}} \sin^{\gamma-1}\alpha\,d\alpha$$

$$= C_{\gamma,1} \int\limits_{\frac{x_1^2+y_1^2-r^2}{2x_1y_1}}^1 (1-t^2)^{\frac{\gamma}{2}-1}\,dt \le C_{\gamma,1} 2^{(\frac{\gamma}{2}-1)_+} \int\limits_{\frac{x_1^2+y_1^2-r^2}{2x_1y_1}}^1 (1-t)^{\frac{\gamma}{2}-1}\,dt$$

$$\le \frac{C_{\gamma,1}}{\gamma} 2^{(\frac{\gamma}{2}-1)_+ +1} \left(1 - \frac{x_1^2+y_1^2-r^2}{2x_1y_1}\right)^{\frac{\gamma}{2}} \le \frac{C_{\gamma,1}}{\gamma} 2^{(\frac{\gamma}{2}-1)_+ +1} \left(\frac{r}{x_1}\right)^{\frac{\gamma}{2}} \left(\frac{r-|x_1-y_1|}{y_1}\right)^{\frac{\gamma}{2}}.$$

Since $\frac{r-|x_1-y_1|}{y_1} \le \frac{r}{x_1}$ for $y_1 \ge x_1$ and $y_1 < x_1$, the inequality $\frac{r-|x_1-y_1|}{y_1} < \frac{r}{x_1}$ is equivalent to $r < x_1$. Hence,

$$T^y \chi_{B(0,r)}(x) \le \frac{C_{\gamma,1}}{\gamma} 2^{(\frac{\gamma}{2}-1)_+ +1} \left(\frac{r}{x_1}\right)^{\gamma}. \tag{16}$$

Therefore, (14) follows from (15) and (16).

We now estimate the measure of the ball $B(x,r)$. Start with the case $x_1 \le r$. We have

$$\mu B(x,r) = \int\limits_{B(x,r)} y_1^{\gamma}\, dy \le \prod_{j=2}^{N} \int\limits_{|y_j|<r} dy_j \int\limits_{\{y_1>0;|x_1-y_1|<r\}} y_1^{\gamma}\, dy_1$$

$$\le (2r)^{N-1} \int\limits_0^{x_1+r} y_1^{\gamma}\, dy_1 = (2r)^{N-1} \frac{(x+r)^{\gamma+1}}{\gamma+1} \le \frac{(2r)^Q}{\gamma+1}.$$

Suppose now that $x_1 > r$. Then

$$\mu B(x,r) \le (2r)^{N-1} \int\limits_{x_1-r}^{x_1+r} y_1^{\gamma}\, dy_1 \le (2r)^N (x_1+r)^{\gamma} = (2r)^Q \frac{x_1^{\gamma}}{r^{\gamma}}.$$

Combining the last two estimates, we get

$$\mu B(x,r) \le (2r)^Q \max\{1, x_1^{\gamma}/r^{\gamma}\}. \tag{17}$$

Estimate $M_{\gamma}^{\alpha} f(x)$ as follows:

$$M_{\gamma}^{\alpha} f(x) \le M_{0,\gamma}^{\alpha} f(x) + M_{1,\gamma}^{\alpha} f(x),$$

where

$$M_{0,\gamma}^{\alpha} f(x) = \sup_{r\le x_1} |B(0,r)|_{\gamma}^{\alpha/Q-1} \int\limits_{B(x,r)} |f(y)| T^y \chi_{B(0,r)}(x) (y')^{\gamma}\, dy,$$

$$M_{1,\gamma}^{\alpha} f(x) = \sup_{r> x_1} |B(0,r)|_{\gamma}^{\alpha/Q-1} \int\limits_{B(x,r)} |f(y)| T^y \chi_{B(0,r)}(x) (y')^{\gamma}\, dy.$$

In the case $x_1 < r$, considering $\mu B(x,r) \le (2r)^Q$, $T^y \chi_{B(0,r)}(x) \le 1$, and $|B(0,r)|_{\gamma} = \omega(N,1,\gamma) r^Q$, we obtain

$$M_{1,\gamma}^{\alpha} f(x) = \sup_{r> x_1} |B(0,r)|_{\gamma}^{\alpha/Q-1} \int\limits_{B(x,r)} |f(y)| T^y \chi_{B(0,r)}(x) (y')^{\gamma}\, dy$$

$$\le \omega(N,1,\gamma)^{\alpha/Q-1} 2^{Q-\alpha} \sup_{r>0} (\mu B(x,r))^{\beta-1} \int\limits_{B(x,r)} |f(y)|\, d\mu(y)$$

$$\le \omega(N,1,\gamma)^{\alpha/Q-1} 2^{Q-\alpha} M_{\mu}^{\beta} f(x).$$

In the case $x_1 \ge r$, since $\mu B(x,r) \le (2r)^Q \frac{x_1^{\gamma}}{r^{\gamma}}$, $T^y \chi_{B(0,r)}(x) \le c_2 r^{\gamma}/x_1^{\gamma}$, and $|B(0,r)|_{\gamma} = \omega(N,1,\gamma) r^Q$, we have

$$M_{0,\gamma}^{\alpha} f(x) \le \sup_{r\le x_1} |B(0,r)|_{\gamma}^{\alpha/Q-1} \int\limits_{B(x,r)} |f(y)| T^y \chi_{B(0,r)}(x) (y')^{\gamma}\, dy$$

$$\le c_2 \omega(N,1,\gamma)^{\alpha/Q-1} 2^{Q-\alpha} \sup_{r>0} (\mu B(x,r))^{\beta-1} \int\limits_{B(x,r)} |f(y)|\, d\mu(y)$$

$$= c_2 \omega(N,1,\gamma)^{\alpha/Q-1} 2^{Q-\alpha} M_{\mu}^{\beta} f(x).$$

Therefore, $M_{\gamma}^{\alpha} f(x) \le M_{0,\gamma}^{\alpha} f(x) + M_{1,\gamma}^{\alpha} f(x) \le c_1 M_{\mu}^{\beta} f(x)$.

Since $X = \mathbb{R}^N_{n,+}$, $\beta = \alpha/Q$, $1 < p < 1/\beta$, and $1/p - 1/q = \beta$ in (12) and (13) and $d\mu(x) = (x')^\gamma\, dx$ satisfies the doubling condition, we find that

$$\|M^\alpha_\gamma f\|_{q,\gamma} \le c_1 \|M^\beta_\mu f\|_{q,\gamma} \le C_2 \|f\|_{p,\gamma},$$

where $C_2 = c_1 C_2'$, and, for $p = 1$ and $1 - 1/q = \beta$,

$$\big|\{x \in \mathbb{R}^N_{n,+} : M^\alpha_\gamma f(x) > \tau\}\big|_\gamma \le \mu\{x \in \mathbb{R}^N_{n,+} : M^\beta_\mu f(x) > \tau/c_1\} \le \bigg(\frac{C_1}{\tau} \int\limits_{\mathbb{R}^N_{n,+}} |f(x)|\, d\mu(x)\bigg)^q,$$

where $C_1 = c_1 C_1'$.

Theorem 1 is proved.

Corollary 1. 1. *Let $f \in L_{1,\gamma}(\mathbb{R}^N_{n,+})$. Then*

$$\big|\{x \in \mathbb{R}^N_{n,+} : M_\gamma f(x) > \tau\}\big|_\gamma \le \frac{C_3}{\tau} \int\limits_{\mathbb{R}^N_{n,+}} |f(x)|(x')^\gamma\, dx \tag{18}$$

for all $\tau > 0$, where the constant C_3 is independent of f.

2. *Let $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ and $1 < p \le \infty$. Then $M_\gamma f(x) \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ and*

$$\|M_\gamma f\|_{p,\gamma} \le C_4 \|f\|_{p,\gamma}, \tag{19}$$

where C_4 is a constant independent of f.

Corollary 2. *If $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ and $1 \le p \le \infty$ then*

$$\lim_{r\to 0} |B(0,r)|^{-1}_\gamma \int\limits_{B(0,r)} T^y f(x)(y')^\gamma\, dy = f(x)$$

for almost all $x \in \mathbb{R}^N_{n,+}$.

Remark 1. Note that for $0 < \alpha < Q$ Theorem 1 is new even in the one-dimensional case. Corollary 1 is proved in [17] for the one-dimensional case, i.e., for $N = n = 1$, and in [7] for the multidimensional case for $n = N \ge 2$ (see [8] for details).

3. Pointwise and Integral Estimates for the B-Riesz Potential

Consider the B-Riesz potential

$$I^\alpha_\gamma f(x) = \int\limits_{\mathbb{R}^N_{n,+}} T^y |x|^{\alpha - Q} f(y)(y')^\gamma\, dy, \quad 0 < \alpha < Q.$$

It is easy to show that if $p \ge \frac{Q}{\alpha}$ then $I^\alpha_\gamma f$ is not defined for all functions $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$.

Let us prove the following theorem which yields a pointwise estimate for the B-Riesz potential $I^\alpha_\gamma f(x)$. For a Riesz potential some estimates of this kind were obtained in [13].

Theorem 2. *Let $0 < \alpha < Q$ and let $f \in L_{p,\gamma}(\mathbb{R}^N_{n,+})$ be a locally summable function. Then for $r > 0$ and $x \in \mathbb{R}^N_{n,+}$ there are constants C_5–C_8 depending only on α, p, n, N, and γ such that*

$$I^\alpha_\gamma |f|(x) \le C_5 \big(r^\alpha M_\gamma f(x) + r^{\alpha - \frac{\lambda}{p}} M^{\lambda/p}_\gamma f(x)\big), \quad 1 \le p < \lambda/\alpha; \tag{20}$$

$$I^\alpha_\gamma |f|(x) \le C_6 \|f\|^{\frac{\alpha p}{Q}}_{p,\gamma} (M_\gamma f(x))^{1 - \frac{\alpha p}{Q}}, \quad 1 \le p < Q/\alpha; \tag{21}$$

$$I^{\alpha\theta}_\gamma f(x) \le C_7 \big(I^\alpha_\gamma f(x)\big)^\theta (M_\gamma f(x))^{1-\theta}, \quad 0 < \theta < 1; \tag{22}$$

$$I^{\alpha\theta}_\gamma f(x) \le C_8 \big(M^\alpha_\gamma f(x)\big)^\theta (M_\gamma f(x))^{1-\theta}, \quad 0 < \theta < 1. \tag{23}$$

Proof. Let $r > 0$ be an arbitrary number. Using the properties of convolution, present $I_\gamma^\alpha |f|(x)$ as

$$I_\gamma^\alpha |f|(x) = \int\limits_{\mathbb{R}^N_{n,+}} |y|^{\alpha-Q} T^y |f(x)| (y')^\gamma \, dy$$
$$= \Bigg(\int\limits_{B(0,r)} + \int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,r)} \Bigg) T^y |f(x)| \, |y|^{\alpha-Q} (y')^\gamma \, dy = J_1(x,r) + J_2(x,r).$$

To prove (20), we first estimate $J_1(x,r)$. Summing over all $j > 0$, we get

$$J_1(x,r) \le \int\limits_{B(0,r)} T^y |f(x)| \, |y|^{\alpha-Q} (y')^\gamma \, dy$$
$$= \sum_{j=1}^{\infty} \int\limits_{B(0,2^{-j+1}r)\setminus B(0,2^{-j}r)} T^y |f(x)| \, |y|^{\alpha-Q} (y')^\gamma \, dy \le c r^\alpha M_\gamma f(x).$$

Therefore,

$$J_1(x,t) \le c t^\alpha M_\gamma f(x). \tag{24}$$

Similarly, estimate $J_2(x,r)$:

$$J_2(x,r) = \int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,r)} |y|^{\alpha-Q} T^y |f(x)| (y')^\gamma \, dy$$
$$\le \sum_{k=0}^{\infty} \int\limits_{B(0,2^{k+1}r)\setminus B(0,2^k r)} |y|^{\alpha-Q} T^y |f(x)| (y')^\gamma \, dy \le c r^{\alpha - \frac{\lambda}{p}} M_\gamma^{\frac{\lambda}{p}} f(x),$$

since $\alpha - \frac{\lambda}{p} < 0$ by assumption.

Thus, (20) is proved. We turn to the proof of (21). Using (1) and the estimate for $J_2(x,t)$ given below, we obtain (21).

Applying Hölder's inequality and (1), we infer

$$J_2(x,t) \le \Bigg(\int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} (T^y |f(x)|)^p (y')^\gamma \, dy \Bigg)^{1/p} \Bigg(\int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} |y|^{(\alpha-Q)p'} (y')^\gamma \, dy \Bigg)^{1/p'}$$
$$\le \|T^y f\|_{p,\gamma} \Bigg(\int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} |y|^{(\alpha-Q)p'} (y')^\gamma \, dy \Bigg)^{1/p'}$$
$$\le \|f\|_{p,\gamma} \Bigg(\int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} |y|^{(\alpha-Q)p'} (y')^\gamma \, dy \Bigg)^{1/p'}.$$

Pass to the spherical coordinates:

$$\left(\int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} |y|^{(\alpha-Q)p'}(y')^\gamma\,dy\right)^{1/p'} = \left(\int\limits_t^\infty \int\limits_{S^{N-1}_+} r^{(\alpha-Q)p'+Q-1}(\theta')^\gamma\,drd\theta\right)^{1/p'}$$

$$= \left(\int\limits_{S^{N-1}_+} (\theta')^\gamma\,d\theta \int\limits_t^\infty r^{(\alpha-Q)p'+Q-1}\,dr\right)^{1/p'}$$

$$= \left(\int\limits_{S^{N-1}_+} (\theta')^\gamma\,d\theta\right)^{1/p'} \left(\int\limits_t^\infty r^{(\alpha-Q)p'+N+\gamma-1}\,dr\right)^{1/p'} = ct^{\alpha-Q+\frac{Q}{p'}} = ct^{\alpha-\frac{Q}{p}}.$$

Therefore,

$$J_2(x,t) \le c\|f\|_{p,\gamma} t^{\alpha-\frac{Q}{p}}. \tag{25}$$

Hence, from (24) and (25)

$$I^\alpha_\gamma |f|(x) \le c(t^\alpha M_\gamma f(x) + \|f\|_{p,\gamma} t^{\alpha-\frac{Q}{p}}).$$

Minimizing by t, for $t = [(M_\gamma f(x))^{-1}\|f\|_{p,\gamma}]^{p/(Q)}$ we obtain

$$I^\alpha_\gamma |f|(x) \le C_6 \|f\|^{\frac{\alpha p}{Q}}_{p,\gamma} (M_\gamma f(x))^{1-\frac{\alpha p}{Q}}.$$

We now prove (22). Consider

$$I^{\alpha\theta}_\gamma f(x) = \int\limits_{\mathbb{R}^N_{n,+}} T^y f(x)|y|^{\alpha\theta-Q}(y')^\gamma\,dy$$

$$= \left(\int\limits_{B(0,t)} + \int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)}\right) T^y f(x)|y|^{\alpha\theta-Q}(y')^\gamma\,dy = I_1(x,t) + I_2(x,t).$$

Estimate $I_2(x,t)$. Note that since, by the hypotheses of the theorem, $0<\theta<1$, we have $\alpha\theta-\alpha<0$ and $|y|^{\alpha\theta-\alpha} \le t^{\alpha\theta-\alpha}$ for all $y \in \mathbb{R}^N_{n,+}\setminus B(0,t)$. Therefore,

$$I_2(x,t) = \int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} T^y f(x)|y|^{\alpha\theta-Q}(y')^\gamma\,dy$$

$$\le t^{\alpha\theta-\alpha} \int\limits_{\mathbb{R}^N_{n,+}\setminus B(0,t)} T^y f(x)|y|^{\alpha-Q}(y')^\gamma\,dy \le t^{\alpha\theta-\alpha} I^\alpha_\gamma f(x). \tag{26}$$

Using (24) and (26), we get

$$I_1(x,t) \le ct^{\alpha\theta} M_\gamma f(x), \tag{27}$$

$$I_2(x,t) \le t^{\alpha\theta-\alpha} I^\alpha_\gamma f(x). \tag{28}$$

Hence, it follows from (27) and (28) that

$$I^{\alpha\theta}_\gamma f(x) \le ct^{\alpha\theta} M_\gamma f(x) + t^{\alpha\theta-\alpha} I^\alpha_\gamma f(x). \tag{29}$$

Minimizing by t, for $t = \left[(M_\gamma f(x))^{-1} I_\gamma^\alpha f(x)\right]^{1/\alpha}$ we find that

$$I_\gamma^{\alpha\theta} f(x) \le C_7 \big(I_\gamma^\alpha f(x)\big)^\theta (M_\gamma f(x))^{1-\theta}.$$

Turn to the proof of (23). Consider $I_2(x,t)$. Summing over all $j > 0$, we obtain

$$I_2(x,t) \le \sum_{j=0}^{\infty} \int\limits_{B(0,2^{j+1}t)\setminus B(0,2^j t)} T^y f(x) |y|^{\alpha\theta - Q} (y')^\gamma \, dy \le \sum_{j=0}^{\infty} (2^j t)^{\alpha\theta - N - |\gamma|} \int\limits_{B(0,2^{j+1}t)} T^y f(x) (y')^\gamma \, dy$$

$$\le 2^{Q-\alpha} t^{\alpha\theta-\alpha} M_\gamma^\alpha f(x) \sum_{j=0}^{\infty} (2^{\alpha\theta-\alpha})^j \le c t^{\alpha\theta-\alpha} M_\gamma^\alpha f(x).$$

Therefore,

$$I_2(x,t) \le c^{\alpha\theta-\alpha} M_\gamma^\alpha f(x). \tag{30}$$

Using (27) and (30), we deduce

$$I_\gamma^{\alpha\theta} f(x) \le t^{\alpha\theta} M_\gamma f(x) + c t^{\alpha\theta-\alpha} M_\gamma^\alpha f(x).$$

Minimizing by t, for $t = \left[(M_\gamma f(x))^{-1} M_\gamma^\alpha f(x)\right]^{1/\alpha}$ we obtain

$$I_\gamma^{\alpha\theta} f(x) \le C_8 \big(M_\gamma^\alpha f(x)\big)^\theta (M_\gamma f(x))^{1-\theta}.$$

Theorem 2 is proved.

Theorem 3. *Let* $0 < \alpha < Q$ *and* $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$.

(a) *If* $1 < p < \frac{\lambda}{\alpha}$, $1 \le r \le \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\lambda} + \frac{\alpha p}{\lambda r}$ *then*

$$\big\|I_\gamma^\alpha f\big\|_{q,\gamma} \le C_9 \big\|M_\gamma^{\frac{\lambda}{p}} f\big\|_{r,\gamma}^{\frac{\alpha p}{\lambda}} \|f\|_{p,\gamma}^{1-\frac{\alpha p}{\lambda}} \tag{31}$$

for all $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ *and all* $M_\gamma^{\lambda/p} f \in L_{r,\gamma}(\mathbb{R}_{n,+}^N)$, *where* C_9 *is a constant independent of* f.

(b) *If* $1 < p < \frac{Q}{\alpha}$ *then there are constants* C_{10} *and* C_{11} *depending only on* α, p, n, N, *and* γ *such that*

$$\big\|I_\gamma^{\alpha\theta} f\big\|_{r,\gamma} \le C_{10} \big\|I_\gamma^\alpha |f|\big\|_{q,\gamma}^\theta \|f\|_{p,\gamma}^{1-\theta}, \tag{32}$$

$$\big\|I_\gamma^{\alpha\theta} f\big\|_{r,\gamma} \le C_{11} \big\|M_\gamma^\alpha f\big\|_{q,\gamma}^\theta \|f\|_{L_{p,\gamma}}^{1-\theta}, \tag{33}$$

where $0 < \theta < 1$, $0 < q \le \infty$, $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$.

Proof. Inserting

$$r = r(x) = \left(\frac{M_\gamma^{\lambda/p} f(x)}{M_\gamma f(x)}\right)^{p/\lambda}$$

in (20), we have

$$\big|I_\gamma^\alpha f(x)\big| \le C_5 \big(M_\gamma^{\frac{\lambda}{p}} f(x)\big)^{\frac{\alpha p}{\lambda}} (M_\gamma f(x))^{1-\frac{\alpha p}{\lambda}} \tag{34}$$

for all $x \in \mathbb{R}_{n,+}^N$. Taking the qth power of the two sides of the inequality, integrating over x, and applying Hölder's inequality to the right-hand side of (34), we obtain

$$\int\limits_{\mathbb{R}_{n,+}^N} \big|I_\gamma^\alpha f(x)\big|^q (x')^\gamma dx \le C_5^q \int\limits_{\mathbb{R}_{n,+}^N} \big(M_\gamma^{\frac{\lambda}{p}} f(x)\big)^{\frac{\alpha pq}{\lambda}} (M_\gamma f(x))^{q-\frac{\alpha pq}{\lambda}} (x')^\gamma \, dx$$

$$\le C_5^q \big\|\big(M_\gamma^{\frac{\lambda}{p}} f\big)^{\frac{\alpha pq}{\lambda}}\big\|_{s',\gamma} \|(M_\gamma f)^{q-\frac{\alpha pq}{\lambda}}\|_{s,\gamma},$$

where $\left(q-\frac{\alpha pq}{\lambda}\right)s=p$, $s'=\frac{s}{s-1}=\frac{\lambda r}{\alpha pq}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\lambda}+\frac{\alpha p}{\lambda r}$. Therefore,

$$\left\|I_\gamma^\alpha f\right\|_{q,\gamma}\le C_5\left\|M_\gamma^{\frac{\lambda}{p}}f\right\|_{r,\gamma}^{\frac{\alpha p}{\lambda}}\|M_\gamma f\|_{p,\gamma}^{\frac{p}{qs}}\le C_9\left\|M_\gamma^{\frac{\lambda}{p}}f\right\|_{r,\gamma}^{\frac{\alpha p}{\lambda}}\|f\|_{p,\gamma}^{\frac{p}{qs}}=C_9\left\|M_\gamma^{\frac{\lambda}{p}}f\right\|_{r,\gamma}^{\frac{\alpha p}{\lambda}}\|f\|_{p,\gamma}^{1-\frac{\alpha p}{\lambda}}.$$

We now prove (32); (33) can be checked analogously.

Consider $\left\|I_\gamma^{\alpha\theta}f\right\|_{r,\gamma}$. From (22) and Hölder's inequality we obtain

$$\left\|I_\gamma^{\alpha\theta}f\right\|_{r,\gamma}\le C_7\left\|\left(I_\gamma^\alpha|f|\right)^\theta(M_\gamma f)^{1-\theta}\right\|_{r,\gamma}\le C_7\left\|\left(I_\gamma^\alpha|f|\right)^\theta\right\|_{r\tau',\gamma}\|(M_\gamma f)^{1-\theta}\|_{r\tau,\gamma}.$$

Put $p=(1-\theta)r\tau$, $q=\theta r\tau'$, where $\tau'=\frac{\tau}{\tau-1}$. Then, obviously, $\frac{1}{r\tau}=\frac{1-\theta}{p}$ and $\frac{1}{r\tau'}=\frac{\theta}{q}$. From the above we have

$$\left\|I_\gamma^{\alpha\theta}f\right\|_{r,\gamma}\le C_7\|I_\gamma^\alpha|f|\,\|_{q,\gamma}^\theta\|M_\gamma f\|_{p,\gamma}^{1-\theta}.$$

The last inequality, (19), and Corollary 1 yield

$$\left\|I_\gamma^{\alpha\theta}f\right\|_{r,\gamma}\le C_{10}\left\|I_\gamma^\alpha|f|\,\right\|_{q,\gamma}^\theta\|f\|_{p,\gamma}^{1-\theta}.$$

Theorem 3 is proved.

We now consider the modified B-Riesz potential

$$\widetilde{I}_\gamma^\alpha f(x)=\int\limits_{\mathbb{R}_{n,+}^N}(T^y|x|^{\alpha-Q}-|y|^{\alpha-Q}\chi_{B*(0,1)}(y))f(y)(y')^\gamma\,dy,$$

where $B^*(0,1)=\mathbb{R}_{n,+}^N\backslash B(0,1)$.

Using Theorems 1 and 2, we can obtain Sobolev's theorem that was proved in [18] for a B-Riesz potential; namely, we can show that I_γ^α is an operator of strong type $(p,q)_\gamma$, $1<p<Q/\alpha$, $1/p-1/q=\alpha/Q$ and of weak type $(1,q)_\gamma$, $1/q=1-\alpha/Q$ (see the case $N=n=1$ in [19]; the case $N\ge 2$ and $n=1$ in [20]; and the case $N=n\ge 2$ in [7,21]).

Theorem 4. *Let $0<\alpha<Q$, $1\le p\le\frac{Q}{\alpha}$.*

(a) If $p=1$ then the condition $1-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for I_γ^α to be bounded from $L_{1,\gamma}\left(\mathbb{R}_{n,+}^N\right)$ to $WL_{q,\gamma}\left(\mathbb{R}_{n,+}^N\right)$.

(b) If $1<p<\frac{Q}{\alpha}$ then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for I_γ^α to be bounded from $L_{p,\gamma}\left(\mathbb{R}_{n,+}^N\right)$ to $L_{q,\gamma}\left(\mathbb{R}_{n,+}^N\right)$.

(c) If $p=\frac{Q}{\alpha}$, $f\in L_{p,\gamma}\left(\mathbb{R}_{n,+}^N\right)$, $\|f\|_{L_{p,\gamma}}=1$, and the support of f belongs to $B(0,r)$ then

$$\int\limits_{B(0,r)}\exp\left(\beta_0\left|I_\gamma^\alpha f(x)\right|^{p'}\right)(x')^\gamma\,dx\le C_0\omega(N,n,\gamma)r^Q,$$

where $\beta_0=\omega(N,n,\gamma)^{-1}(pC_{\gamma,n})^{-p'}$, and $C_0=C_0(N,\alpha,\gamma)$ is a constant depending only on N, α, and γ.

(d) If $p=\frac{Q}{\alpha}$ and $f\in L_{p,\gamma}\left(\mathbb{R}_{n,+}^N\right)$ then $\widetilde{I}_\gamma^\alpha f\in BMO_\gamma\left(\mathbb{R}_{n,+}^N\right)$ and

$$\left\|\widetilde{I}_\gamma^\alpha f\right\|_{BMO_\gamma}\le C_{12}\|f\|_{p,\gamma},$$

where $C_{12}>0$ is a constant independent of f.

If the integral $I_\gamma^\alpha f$ exists almost everywhere then $I_\gamma^\alpha f\in BMO_\gamma(\mathbb{R}_{n,+}^N)$ and

$$\left\|I_\gamma^\alpha f\right\|_{BMO_\gamma}\le C_{13}\|f\|_{p,\gamma},$$

where $C_{13}>0$ is a constant independent of f.

Remark 2. The parts (a) and (b) of Theorem 4 were proved in [11, 14, 21, 22] by different methods (also see [18]). In the case $n = 1$ the parts (a) and (b) of Theorem 4 were proved in [20], and in the case $n = N$, they were proved in [7] (also see [8]).

Proof. First of all, we prove (c). Put $A = |B(0,r)|_\gamma$. Applying Lemma 1, an analog of the O'Neil inequality for generalized convolutions in [11, 14], and Lemma 2, we have

$$\begin{aligned}\left(I_\gamma^\alpha f\right)_\gamma^*(t) &\le \left(I_\gamma^\alpha f\right)_\gamma^{**}(t)\\ &\le C_{\gamma,n}\left(f_\gamma^{**}(t)\int_0^t(|\cdot|^{\alpha-N-\gamma})_\gamma^{**}(s)ds+\int_t^\infty f_\gamma^*(s)(|\cdot|^{\alpha-N-\gamma})_\gamma^{**}(s)\,ds\right)\\ &= C_{\gamma,n}\left(f_\gamma^{**}(t)\int_0^t\frac{Q}{\alpha}\left(\frac{\omega(N,n,\gamma)}{s}\right)^{\frac{Q-\alpha}{Q}}ds+\int_t^A f_\gamma^*(s)\frac{Q}{\alpha}\left(\frac{\omega(N,n,\gamma)}{s}\right)^{\frac{Q-\alpha}{Q}}ds\right).\end{aligned}\tag{35}$$

Since $p = \frac{Q}{\alpha}$, $p' = \frac{p}{p-1} = \frac{Q}{Q-\alpha}$, and $\frac{1}{p'} = \frac{Q-\alpha}{Q}$, we find that

$$\left(I_\gamma^\alpha f\right)_\gamma^*(t) \le C_{\gamma,n}p\,\omega(N,n,\gamma)^{\frac{1}{p'}}\left(pt^{-\frac{1}{p'}}\int_0^t f_\gamma^*(s)\,ds+\int_t^A f_\gamma^*(s)s^{-\frac{1}{p'}}\,ds\right).\tag{36}$$

Put

$$a(s,t)=\begin{cases}1, & 0<s<t,\\ pe^{(t-s)/p'}, & t<s<\infty,\\ 0, & -\infty<s\le 0,\end{cases}\qquad \phi(s)=A^{1/p}f_\gamma^*(Ae^{-s})e^{-s/p}.$$

Then

$$\sup_{t>0}\left(\int_{-\infty}^0+\int_t^\infty a(s,t)^{p'}\,ds\right)^{1/p'}=\sup_{t>0}\left(\int_t^\infty(pe^{(t-s)/p'})^{p'}\,ds\right)^{1/p'}=p<\infty,$$

$$\begin{aligned}\int_{-\infty}^\infty \phi(s)^p\,ds &= \int_{-\infty}^\infty Af_\gamma^*(Ae^{-s})^pe^{-s}\,ds=\int_0^\infty f_\gamma^*(t)^p\,dt\\ &=\int_0^A f_\gamma^*(t)^p\,dt=\int_{B(0,r)}|f(x)|^p(x')^\gamma\,dx=1.\end{aligned}$$

Since $a(s,t)$ and $\phi(s)$ satisfy (3)–(5), by Lemma 3, there is a constant C_0, independent of p such that

$$\int_0^\infty e^{-F(t)}\,dt\le C_0,$$

where $F(t)=t-\left(\int_{-\infty}^\infty a(s,t)\phi(s)\,ds\right)^{p'}$. Inserting the values of $a(s,t)$ and $\phi(s)$, we get

$$\begin{aligned}F(t)=t-\left(\int_0^t\phi(s)\,ds+\int_t^\infty pe^{(t-s)/p'}\phi(s)\,ds\right)^{p'}&=t-\left(\int_0^t A^{1/p}f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds\right.\\ &\left.+\left(\int_t^\infty pe^{(t-s)/p'}A^{1/p}f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds\right)^{p'}.\end{aligned}$$

Changing variables, we obtain

$$F\left(\log\frac{A}{t}\right) = \log\frac{A}{t} - \Bigg(\int_0^{\log\frac{A}{t}} A^{1/p} f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds$$

$$+\int_{\log\frac{A}{t}}^{\infty} pe^{(\log\frac{A}{t}-s)/p'} A^{1/p} f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds\Bigg)^{p'} = \log\frac{A}{t} - (I_1+I_2)^{p'}.$$

Estimate I_1 and I_2:

$$I_1 = \int_0^{\log\frac{A}{t}} A^{1/p} f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds = \int_t^A f_\gamma^*(\tau)\tau^{-\frac{1}{p'}}\,d\tau,$$

$$I_2 = \int_{\log\frac{A}{t}}^{\infty} pe^{(\log\frac{A}{t}-s)/p'} A^{1/p} f_\gamma^*(Ae^{-s})e^{-\frac{s}{p}}\,ds = \int_{\log\frac{A}{t}}^{\infty} e^{\frac{\log\frac{A}{t}}{p'}} e^{-\frac{s}{p'}} e^{-\frac{s}{p}} A^{1/p} f_\gamma^*(Ae^{-s})\,ds$$

$$= \int_{\log\frac{A}{t}}^{\infty} pAt^{-\frac{1}{p'}} e^{-s} f_\gamma^*(Ae^{-s})\,ds = pt^{-\frac{1}{p'}}\int_0^t f_\gamma^*(\tau)\,d\tau.$$

Inserting estimates for I_1 and I_2, we have

$$F\left(\log\frac{A}{t}\right) = \log\frac{A}{t} - \left(pt^{-\frac{1}{p'}}\int_0^t f_\gamma^*(\tau)\,d\tau + \int_t^A f_\gamma^*(\tau)\tau^{-\frac{1}{p'}}\,d\tau\right)^{p'}. \tag{37}$$

Combining (35)–(37), we obtain

$$C_0 \ge \int_0^\infty e^{-F(t)}\,dt = \int_0^A t^{-1}e^{-F(\log\frac{A}{t})}\,dt$$

$$= \int_0^A t^{-1}\exp\left\{\left(pt^{-\frac{1}{p'}}\int_0^t f_\gamma^*(\tau)\,d\tau + \int_t^A f_\gamma^*(\tau)\tau^{-\frac{1}{p'}}\,d\tau\right)^{p'} - \log\frac{A}{t}\right\}dt$$

$$= \frac{1}{A}\int_0^A \exp\left\{\left(pt^{-\frac{1}{p'}}\int_0^t f_\gamma^*(\tau)\,d\tau + \int_t^A f_\gamma^*(\tau)\tau^{-\frac{1}{p'}}\,d\tau\right)^{p'}\right\}dt$$

$$\ge \frac{1}{A}\int_0^A \exp\left\{\omega(N,n,\gamma)^{-1}\left[\frac{\left(I_\gamma^\alpha f\right)_\gamma^*(t)}{C_{\gamma,n}p}\right]^{p'}\right\}dt = \frac{1}{A}\int_{B(0,r)} \exp\left(\beta_0\left|I_\gamma^\alpha f(x)\right|^{p'}\right)(x')^\gamma\,dx.$$

Therefore,

$$\frac{1}{A}\int_{B(0,r)} \exp\left(\beta_0\left|I_\gamma^\alpha f(x)\right|^{p'}\right)(x')^\gamma\,dx \le C_0.$$

The part (c) of Theorem 4 is proved.

Note that the proof of the part (d) of Theorem 4 is analogous to that of Theorem 4 in [8].

The proof of Theorem 4 is complete.

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