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ON THE REARRANGEMENT ESTIMATES AND THE BOUNDEDNESS OF THE GENERALIZED FRACTIONAL INTEGRALS ASSOCIATED WITH THE LAPLACE–BESSEL DIFFERENTIAL OPERATOR

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Abstract. We introduce the generalized fractional integrals (generalized B-fractional integrals) generated by the Δ_B Laplace–Bessel differential operator and give some results for them. We obtain O'Neil type inequalities for the B-convolutions and give pointwise rearrangement estimates of the generalized B-fractional integrals. Then we get the $L_{p,\gamma}$-boundedness of the generalized B-convolution operator, the generalized B-Riesz potential and the generalized fractional B-maximal function. Finally, we prove a sharp pointwise estimate of the nonincreasing rearrangement of the generalized fractional B-maximal function.

1. Introduction and statement of main results

The potential type integral operators have an important place in the theory of harmonic analysis and partial differential equations. The potentials and related topics associated with the Bessel differential expansion

$$B = (B_1, \dots, B_k), \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_i > 0,\ i = 1, \dots, k$$

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have been research areas for many mathematicians such as B. M. Levitan [11], B. Muckenhoupt and E. M. Stein [15], I. A. Kipriyanov [8], L. N. Lyakhov [13], K. Stempak [19], A. D. Gadjiev and I. A. Aliev [1], A. Serbetci and I. Ekincioglu [17], V. S. Guliyev [6] and others.

In this paper we realize some estimates of the generalized B-fractional integrals generated by the generalized shift operator of the form (see [9, 11, 13])

$$T^y f(x) = C_{k,\gamma} \int_0^\pi \dots \int_0^\pi f\big((x',y')_\alpha, x''-y''\big)\, d\nu(\alpha),$$

where

$$C_{k,\gamma} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\big((\gamma_i+1)/2\big)}{\Gamma(\gamma_i/2)}, \quad (x_i,y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2},$$
$$1 \leqq i \leqq k,$$

$$(x',y')_\alpha = \big((x_1,y_1)_{\alpha_1}, \dots, (x_k,y_k)_{\alpha_k}\big) \quad \text{and} \quad d\nu(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i,$$
$$1 \leqq k \leqq n.$$

Note that the generalized shift operator T^y is closely related to the Δ_B Laplace–Bessel differential operator

$$\Delta_B = \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2},$$

and generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\, T^y g(x)\, (y')^\gamma \, dy.$$

Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, \dots, x_n)$ defined by the inequalities $x_1 > 0, \dots, x_k > 0$, $1 \leqq k \leqq n$. We write $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, and define $B(x,r) = \big\{ y \in \mathbb{R}^n_{k,+};\, |x-y| < r \big\}$, $\mathbb{S}^{n-1}_{k,+} = \big\{ x \in \mathbb{R}^n_{k,+} : |x| = 1 \big\}$, $(x')^\gamma = x_1^{\gamma_1} \cdot \dots \cdot x_k^{\gamma_k}$, where $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + \dots + \gamma_k$. For any measurable set $E \subset \mathbb{R}^n_{k,+}$, define $|E|_\gamma = \int_E (x')^\gamma \, dx$, then $\big|B(0,r)\big|_\gamma = \omega(n,k,\gamma)\, r^Q$, where $\omega(n,k,\gamma) = \big|B(0,1)\big|_\gamma$, $Q = n + |\gamma|$.

Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all classes of measurable functions f with the finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_{k,+}} \big| f(x) \big|^p (x')^\gamma \, dx \right)^{1/p}, \quad 1 \leqq p < \infty.$$

If $p = \infty$, we assume

$$L_{\infty,\gamma} \equiv L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) = \Big\{ f : \, \|f\|_{L_{\infty,\gamma}} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} \big| f(x) \big| < \infty \Big\}.$$

In order to state the main results we define the generalized fractional B-maximal function by

$$M_{\Omega,\alpha,\gamma} f(x) = \sup_{r>0} \frac{1}{r^{Q-\alpha}} \int_{B(0,r)} \big| \Omega(y) \big| T^y \big| f(x) \big| (y')^\gamma \, dy, \tag{1}$$

the generalized B-Riesz potential by

$$I_{\Omega,\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} \frac{\Omega(y)}{|y|^{Q-\alpha}} T^y f(x) (y')^\gamma \, dy, \tag{2}$$

and the generalized B-convolution operator by

$$(K_\alpha \otimes f)(x) = \int_{\mathbb{R}^n_{k,+}} K_\alpha(y) T^y f(x) (y')^\gamma \, dy, \tag{3}$$

where $0 \leqq \alpha < Q$, $Q = n + |\gamma|$, $\Omega \in L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})$, $s \geqq 1$ and K_α belongs to the weak $L_{p,\gamma}$ space $WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$.

The aim of this paper is to obtain O'Neil type inequalities for the B-convolutions, and $L_{p,\gamma}$-boundedness for $M_{\Omega,\alpha,\gamma}$, $I_{\Omega,\alpha,\gamma}$ and $K_\alpha \otimes f$. In the case $\Omega \equiv 1$, V. S. Guliyev [6] has firstly introduced and investigated the $(L_{p,\gamma}, L_{q,\gamma})$-boundedness, $1 < p \leqq q < \infty$ and $1/p - 1/q = \alpha/Q$, of the fractional B-maximal function $M_{\alpha,\gamma}$, and I. A. Aliev and A. D. Gadjiev [1], L. N. Lyakhov [13] and V. S. Guliyev [6] have shown the $(L_{p,\gamma}, L_{q,\gamma})$-boundedness, $1 < p < q < \infty$ and $1/p - 1/q = \alpha/Q$, of the B-Riesz potential $I_{\alpha,\gamma}$ recently.

The following three theorems are our main results. In Theorem 1 we get the O'Neil type inequality for the B-convolutions. In Theorem 2 we obtain a pointwise rearrangement estimate of the generalized B-convolution operator $K_\alpha \otimes f$. Finally, in Theorem 3 we show the validity of a Sobolev type inequality for $K_\alpha \otimes f$.

Theorem 1. *Let f, g be two positive measurable functions on $\mathbb{R}^n_{k,+}$. Then for all $t > 0$ the following inequality holds:*

$$(f \otimes g)^{**}_\gamma(t) \leqq C_{k,\gamma}\biggl(f^{**}_\gamma(t)\int_0^t g^{**}_\gamma(u)\,du + \int_t^\infty f^*_\gamma(u)g^{**}_\gamma(u)\,du\biggr). \tag{4}$$

Theorem 2. *Let $K_\alpha \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, $0 < \alpha < Q$. Then*

$$(K_\alpha \otimes f)^*_\gamma(t) \leqq (K_\alpha \otimes f)^{**}_\gamma(t) \tag{5}$$

$$\leqq C_1\biggl(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,ds + \int_t^\infty s^{\alpha/Q-1}f^*_\gamma(s)\,ds\biggr),$$

where $C_1 = C_{k,\gamma}(Q/\alpha)^2\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}}$.

Theorem 3. *Let $0 < \alpha < Q$, $K_\alpha \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, and $1 \leqq p \leqq Q/\alpha$.*

1) *If $1 < p < Q/\alpha$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p - 1/q = \alpha/Q$, then $K_\alpha \otimes f \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|K_\alpha \otimes f\|_{L_{q,\gamma}} \leqq C_1\,K(p,q)\,\|f\|_{L_{p,\gamma}},$$

where

$$K(p,q) = \bigl(p^{1/q}q^{1/p'} + (p')^{1/q}(q')^{1/p'}\bigr), \quad p' = p/(p-1).$$

2) *If $p = 1$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $1 - 1/q = \alpha/Q$, then*

$$K_\alpha \otimes f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|K_\alpha \otimes f\|_{WL_{q,\gamma}} \leqq 2C_1\|f\|_{L_{p,\gamma}}.$$

3) *If $p = Q/\alpha$, K_α is homogeneous of degree $\alpha - Q$ on $\mathbb{R}^n_{k,+}$, and $f \in L_{p,1,\gamma}(\mathbb{R}^n_{k,+})$, then $K_\alpha \otimes f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|K_\alpha \otimes f\|_{L_{\infty,\gamma}} \leqq 2C_1\|f\|_{L_{p,1,\gamma}},$$

where $L_{p,1,\gamma}(\mathbb{R}^n_{k,+})$ is the Lorentz spaces with finite norm

$$\|f\|_{L_{p,1,\gamma}} = \bigl\|\,f^*_\gamma(t)t^{-1/p'}\bigr\|_{L_1(0,\infty)}.$$

Remark 1. Note that, for $K_\alpha(x) = |x|^{\alpha-Q}$ and $p \geqq Q/\alpha$, there exists a function $f \in L_{p,\gamma}$ such that $(K_\alpha \otimes f)(x) = \infty$ for all $x \in \mathbb{R}^n_{k,+}$.

2. Auxiliary lemmas

For the generalized shift operator T^y the following two lemmas are valid.

LEMMA 1 [12]. *Let $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leqq p \leqq \infty$. Then for all $y \in \mathbb{R}^n_{k,+}$*

$$\left\| T^y f(\cdot) \right\|_{L_{p,\gamma}} \leqq \|f\|_{L_{p,\gamma}}. \tag{6}$$

LEMMA 2. *For any measurable set $\mathcal{A} = (\mathcal{A}', \mathcal{A}'') \subset \mathbb{R}^n_{k,+}$, $\mathcal{A}' = \mathcal{A}_1 \times \ldots \times \mathcal{A}_k \subset (0,\infty)^k$, $\mathcal{A}'' \subset \mathbb{R}^{n-k}$ and for any $y \in \mathbb{R}^n_{k,+}$ the following equality holds:*

$$\int_{\mathcal{A}} T^y g(x)(x')^{\gamma}\,dx = C_{k,\gamma} \int_{(y,0)+\overline{\mathcal{A}}} g\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\Big)\, d\mu(z, \overline{z'}), \tag{7}$$

where $(x,0) = (x, \underbrace{0,\ldots,0}_{k\text{-times}})$, $\overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k)$, $d\mu(z,\overline{z'}) = \overline{z'}^{\gamma-1} dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1 \cdots d\overline{z}_k$, $\overline{z'}^{\gamma-1} = \overline{z}_1^{\gamma_1 - 1} \cdots \overline{z}_k^{\gamma_k - 1}$, $(z, \overline{z'}) \in \mathbb{R}^n_{k,+} \times (0,\infty)^k$, $m_i = \sup \mathcal{A}_i$, $i = 1,\ldots,k$, $\overline{\mathcal{A}} = \big((-m_1, m_1) \times [0, m_1) \times \ldots \times (-m_k, m_k) \times [0, m_k)\big) \times \mathcal{A}''$.

The proof of Lemma 2 is straightforward after applying the substitutions

$$\begin{cases} z'' = x'', \quad z_i = x_i \cos\alpha_i, \quad \overline{z_i} = x_i \sin\alpha_i, \quad 0 \leqq \alpha_i < \pi, \quad i = 1, \ldots, k, \\ \overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k), \quad (z, \overline{z'}) \in \mathbb{R}^n_{k,+} \times (0,\infty)^k. \end{cases} \tag{8}$$

We need the following two generalized Hardy inequalities (see [14]) which are to be used in the proof of Theorem 3.

LEMMA 3. *Let $1 \leqq p \leqq q \leqq \infty$ and let v and w be two functions measurable and positive a.e. on $(0,\infty)$. Then there exists a constant C independent of φ such that*

$$\left(\int_0^\infty \left(\int_0^t \varphi(s)\,ds \right)^q w(t)\,dt \right)^{1/q} \leqq C \left(\int_0^\infty \varphi(t)^p v(t)\,dt \right)^{1/p}, \tag{9}$$

if and only if

$$K = \sup_{r>0} \left(\int_r^\infty w(t)\,dt \right)^{1/q} \left(\int_0^r v(t)^{1-p'}\,dt \right)^{1/p'} < \infty, \tag{10}$$

where $p + p' = pp'$. Moreover, if C is the best constant in (9) and K is defined by (10), then

$$K \leqq C \leqq k(p,q)K. \tag{11}$$

Here the constant $k(p,q)$ *in* (11) *can be written in various forms. For example (see* [16]*)*

$$k(p,q)=p^{1/q}(p')^{1/p'} \quad or \quad k(p,q)=q^{1/q}(q')^{1/p'} \quad or$$

$$k(p,q)=\left(1+q/p'\right)^{1/q}\left(1+p'/q\right)^{1/p'}.$$

LEMMA 4. *Let* $1\leqq p\leqq q\leqq\infty$ *and let* v *and* w *be two functions measurable and positive a.e. on* $(0,\infty)$*. Then there exists a constant* C *independent of* φ *such that*

$$\left(\int_0^\infty\left(\int_t^\infty\varphi(s)\,ds\right)^q w(t)\,dt\right)^{1/q}\leqq C\left(\int_0^\infty\varphi(t)^p v(t)\,dt\right)^{1/p} \tag{12}$$

if and only if

$$K_1=\sup_{r>0}\left(\int_0^r w(t)\,dt\right)^{1/q}\left(\int_r^\infty v(t)^{1-p'}\,dt\right)^{1/p'}<\infty.$$

Moreover, the best constant C *in* (12) *satisfies the inequalities* $K_1\leqq C\leqq k(p,q)K_1$.

Now we define the γ-rearrangement of a measurable function. Let $f:\mathbb{R}^n_{k,+}\to\mathbb{R}$ be a measurable function. Then the γ-rearrangement of f in decreasing order is defined by

$$f^*_\gamma(t)=\inf\left\{s>0:\ f_{*,\gamma}(s)\leqq t\right\},\quad \forall t\in[0,\infty),$$

where $f_{*,\gamma}$ is the γ-distribution function of f defined by

$$f_{*,\gamma}(s)=\left|\left\{x\in\mathbb{R}^n_{k,+}:\ \left|f(x)\right|>s\right\}\right|_\gamma.$$

Some properties of γ-rearrangements of functions are given as follows (see [7, 18]):

1) if $0<p<\infty$, then

$$\int_{\mathbb{R}^n_{k,+}}\left|f(x)\right|^p(x')^\gamma\,dx=\int_0^\infty\left(f^*_\gamma(t)\right)^p\,dt; \tag{13}$$

2) for any $t>0$,

$$\sup_{|E|_\gamma=t}\int_E\left|f(x)\right|(x')^\gamma\,dx=\int_0^t f^*_\gamma(s)\,ds; \tag{14}$$

3)

$$\int_{\mathbb{R}^n_{k,+}} \big| f(x)g(x)\big| (x')^{\gamma}\, dx \leqq \int_0^{\infty} f^*_{\gamma}(t) g^*_{\gamma}(t)\, dt. \tag{15}$$

4)

$$f^*_{\gamma}\big(f_{*,\gamma}(s)\big) \leqq s, \quad (f_{*,\gamma}(s) < \infty); \quad f_{*,\gamma}\big(f^*_{\gamma}(t)\big) \leqq t, \quad (f^*_{\gamma}(t) < \infty); \tag{16}$$

Denote by $WL_{p,\gamma} \equiv WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ space of all measurable functions f with the finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t^{1/p} f^*_{\gamma}(t) < \infty, \quad 1 \leqq p < \infty.$$

The function f^{**} on $(0, \infty)$ is defined by $f^{**}(t) = \frac{1}{t} \int\limits_0^t f^*(s)\, ds$, $t > 0$, and the following inequality holds (see [7]):

$$(f+g)^{**}_{\gamma}(t) \leqq f^{**}_{\gamma}(t) + g^{**}_{\gamma}(t).$$

In the following lemma we give a relation between the generalized shift operator T^y and the γ-rearrangement of the function f.

LEMMA 5. *For any measurable set $\mathcal{A} \subset \mathbb{R}^n_{k,+}$ and $y \in \mathbb{R}^n_{k,+}$ the following equality holds:*

$$\sup_{|\mathcal{A}|_{\gamma}=t} \int_{\mathcal{A}} T^y \big| f(x)\big| (x')^{\gamma}\, dx = C_{k,\gamma} \int_0^t f^*_{\gamma}(s)\, ds. \tag{17}$$

PROOF. By Lemma 2 we have

$$\int_{\mathcal{A}} T^y \big| f(x)\big| (x')^{\gamma}\, dx = C_{k,\gamma} \int_{(y,0)+\overline{\mathcal{A}}} \big|\overline{f}(z, \overline{z'})\big|\, d\mu(z, \overline{z'}), \tag{18}$$

where $\overline{f}(z, \overline{z'}) = f\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \dots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\Big)$. For the function $\overline{f}(z, \overline{z'})$ the analogue of the equality (14) is valid (see, [7]):

$$\sup_{\mu(\overline{\mathcal{A}})=t} \int_{\overline{\mathcal{A}}} \big|\overline{f}(z, \overline{z'})\big|\, d\mu(z, \overline{z'}) = \int_0^t (\overline{f})^*_{\mu}(s)\, ds, \tag{19}$$

where $(\overline{f})^*_{\mu}(s) = \inf\Big\{t > 0 : \mu\Big(\big\{(z, \overline{z'}) : \big|\overline{f}(z, \overline{z'})\big| > t\big\}\Big) \leqq s\Big\}$.

Note that $\mu\big((y,0)+\overline{\mathcal{A}}\big)=|\mathcal{A}|_\gamma$ and $(\overline{f})^*_\mu(s)=f^*_\gamma(s)$. From the equalities (18) and (19) we have

$$\sup_{|\mathcal{A}|_\gamma=t}\int_{\mathcal{A}} T^y\big|f(x)\big|(x')^\gamma\,dx=C_{k,\gamma}\sup_{\mu(\overline{\mathcal{A}})=t}\int_{(y,0)+\overline{\mathcal{A}}}\big|\overline{f}(z,\overline{z'})\big|\,d\mu(z,\overline{z'})$$

$$=C_{k,\gamma}\int_0^t(\overline{f})^*_\mu(s)\,ds=C_{k,\gamma}\int_0^t f^*_\gamma(s)\,ds. \qquad \square$$

LEMMA 6. *Suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$, $\Omega\in L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})$, $1\leqq s\leqq\infty$, *and*

$$g(x)=\frac{\Omega(x)}{|x|^{Q/s}}.$$

Then $g\in WL_{s,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|g\|_{WL_{s,\gamma}}=Q^{-1/s}\|\Omega\|_{L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})}. \tag{20}$$

PROOF. It is not hard to see that

$$g_{*,\gamma}(t)=Q^{-1}\|\Omega\|^s_{L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})}\,t^{-s},$$

$$g^*_\gamma(t)=(Qt)^{-1/s}\|\Omega\|_{L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})},\quad\text{and}\quad g^{**}_\gamma(t)=s'\quad g^*_\gamma(t).$$

Then $g\in WL_{s,\gamma}$ and the equality (20) is valid. $\square$

REMARK 2. From Lemma 6 it can be easily seen that for $0<\alpha<Q$,

$$|x|^{\alpha-Q}\in WL_{Q/(Q-\alpha),\gamma}\quad\text{and}\quad\big\||x|^{\alpha-Q}\big\|_{WL_{Q/(Q-\alpha),\gamma}}=\omega(n,k,\gamma)^{1-\alpha/Q},$$

$$\frac{\Omega(x)}{|x|^{Q-\alpha}}\in WL_{Q/(Q-\alpha),\gamma}$$

and

$$\left\|\frac{\Omega(x)}{|x|^{Q-\alpha}}\right\|_{WL_{Q/(Q-\alpha),\gamma}}=Q^{\alpha/Q-1}\|\Omega\|^{Q/(Q-\alpha)}_{L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})}.$$

Note that the inequality

$$M_{\Omega,\alpha,\gamma}f(x)\leqq I_{|\Omega|,\alpha,\gamma}\big(|f|\big)(x) \tag{21}$$

is valid. Indeed, for all $r > 0$ we have

$$I_{|\Omega|,\alpha,\gamma}\big(|f|\big)(x) \geqq \int_{B(0,r)} \frac{\big|\Omega(y)\big|}{|y|^{Q-\alpha}} T^y \big|f(x)\big| (y')^\gamma \, dy$$

$$\geqq \frac{1}{r^{Q-\alpha}} \int_{B(0,r)} \big|\Omega(y)\big| T^y \big|f(x)\big| (y')^\gamma \, dy.$$

Taking supremum over all $r > 0$, we get (21).

3. An O'Neil type inequality for B-convolutions

Proof of Theorem 1. Note that the methods of proof used here are closer to that in [10]. We choose a measurable set E_t, $t > 0$, such that

$$\Big\{x \in \mathbb{R}^n_{k,+} : \big|f(x)\big| > f^*_\gamma(t)\Big\} \subset E_t \subset \Big\{x \in \mathbb{R}^n_{k,+} : \big|f(x)\big| \geqq f^*_\gamma(t)\Big\}.$$

Set

$$f_1(x) = \big(f(x) - f^*_\gamma(t)\big)\chi_{E_t}(x), \qquad f_2(x) = f(x) - f_1(x).$$

For any measurable set A in $\mathbb{R}^n_{k,+}$ with measure $|A|_\gamma = t$, we have

$$\int_A (g \otimes f_1)(x)(x')^\gamma \, dx = \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma \, dy \int_A T^y g(x)(x')^\gamma \, dx.$$

Hence from Lemma 2, we obtain

$$\int_A (g \otimes f_1)(x)(x')^\gamma \, dx \leqq C_{k,\gamma} \int_0^t g^*_\gamma(u)\, du \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma \, dy$$

$$\leqq C_{k,\gamma} \int_0^t g^{**}_\gamma(u)\, du \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma \, dy$$

$$= C_{k,\gamma}\bigg(\int_{E_t} f(y)(y')^\gamma \, dy - t f^*_\gamma(t)\bigg) \int_0^t g^{**}_\gamma(u)\, du.$$

Thus from (14) we have

$$(g \otimes f_1)^{**}_\gamma(t) = \frac{1}{t} \sup_{|A|_\gamma = t} \int_A (g \otimes f_1)_\gamma(x)(x')^\gamma \, dx$$

$$\leqq C_{k,\gamma}\big(f^{**}_\gamma(t) - f^*_\gamma(t)\big) \int_0^t g^{**}_\gamma(u)\, du.$$

Next, estimate $(g\otimes f_2)^{**}_\gamma(t)$. By using Lemma 5 and equality (14), we get

$$(22)\qquad \big(T^{\cdot}g(x)\big)^*_\gamma(s)\leqq\big(T^{\cdot}g(x)\big)^{**}_\gamma(s)$$

$$=\frac{1}{s}\sup_{|A|_\gamma=s}\int_A T^y g(x)(y')^\gamma\,dy=C_{k,\gamma}g^{**}_\gamma(s).$$

Hence from (15) we obtain

$$(g\otimes f_2)(x)\leqq\int_0^\infty(f_2)^*_\gamma(u)\big(T^{\cdot}g(x)\big)^*_\gamma(u)\,du\leqq C_{k,\gamma}\int_0^\infty(f_2)^*_\gamma(u)g^{**}_\gamma(u)\,du$$

$$=C_{k,\gamma}\bigg(f^*_\gamma(t)\int_0^t g^{**}_\gamma(u)\,du+\int_t^\infty f^*_\gamma(u)g^{**}_\gamma(u)\,du\bigg).$$

Finally, from (14) we get

$$(g\otimes f_2)^{**}_\gamma(t)\leqq C_{k,\gamma}\bigg(f^*_\gamma(t)\int_0^t g^{**}_\gamma(u)\,du+\int_t^\infty f^*_\gamma(u)g^{**}_\gamma(u)\,du\bigg).\qquad\square$$

Proof of Theorem 2. Let $K_\alpha\in WL_{Q/(Q-\alpha),\gamma}$. Then we have

$$(K_\alpha)^*_\gamma(t)\leqq\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}}t^{\alpha/Q-1},$$

and

$$(K_\alpha)^{**}_\gamma(t)\leqq\frac{Q}{\alpha}\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}}t^{\alpha/Q-1}.$$

By using inequality (4) we get the inequality (5). $\square$

Corollary 1. *Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega\in L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})$, $0<\alpha<Q$. Then for the generalized B-Riesz potential the following inequalities hold:*

$$\big(I_{\Omega,\alpha,\gamma}f\big)^*_\gamma(t)\leqq\big(I_{\Omega,\alpha,\gamma}f\big)^{**}_\gamma(t)$$

$$\leqq C_2\bigg(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,ds+\int_t^\infty s^{\alpha/Q-1}f^*_\gamma(s)\,ds\bigg),$$

where $C_2=C_{k,\gamma}\,\alpha^{-2}\,Q^{1+\alpha/Q}\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})}$,

$$\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})}=\bigg(\int_{\mathbb{S}^{n-1}_{k,+}}\big|\Omega(x)\big|^{Q/(Q-\alpha)}\,d\sigma(x)\bigg)^{(Q-\alpha)/Q}.$$

From Corollary 1 and inequality (21) we get the following rearrangement inequalities for the generalized fractional B-maximal function $M_{\Omega,\alpha,\gamma}f$:

COROLLARY 2. *Suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})$, $0 < \alpha < Q$. *Then the following inequalities hold:*

$$\big(M_{\Omega,\alpha,\gamma} f\big)^*_\gamma(t) \leqq \big(M_{\Omega,\alpha,\gamma} f\big)^{**}_\gamma(t)$$

$$\leqq C_2\bigg(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,ds + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,ds\bigg).$$

COROLLARY 3. *For the B-Riesz potential*

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha-Q} f(y)(y')^\gamma\,dy, \quad 0 < \alpha < Q,$$

the following inequalities hold:

$$\big(I_{\alpha,\gamma} f\big)^*_\gamma(t) \leqq \big(I_{\alpha,\gamma} f\big)^{**}_\gamma(t) \leqq C_3\bigg(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,ds + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,ds\bigg),$$

where $C_3 = C_{k,\gamma}(Q/\alpha)^2\omega(n,k,\gamma)^{(Q-\alpha)/Q}$.

4. The boundedness of the generalized B-fractional integrals based on rearrangement estimates

PROOF OF THEOREM 3. 1) Let $1 < p < Q/\alpha$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p - 1/q = \alpha/Q$. By using the inequalities (5) and (13) we get

$$\|K_\alpha \otimes f\|_{L_{q,\gamma}} = \big\|(K_\alpha \otimes f)^*_\gamma\big\|_{L_q(0,\infty)}$$

$$\leqq C_1\bigg(\int_0^\infty t^{q(\alpha/Q-1)}\bigg(\int_0^t f^*_\gamma(s)\,ds\bigg)^q dt\bigg)^{1/q}$$

$$+ C_1\bigg(\int_0^\infty \bigg(\int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,ds\bigg)^q dt\bigg)^{1/q}.$$

From Lemma 3, for the validity of the inequality

$$\bigg(\int_0^\infty t^{q(\alpha/Q-1)}\bigg(\int_0^t f^*_\gamma(s)\,ds\bigg)^q dt\bigg)^{1/q} \leqq C_3\bigg(\int_0^\infty f^*_\gamma(t)^p\,dt\bigg)^{1/p}$$

the necessary and sufficient condition is

$$\sup_{t>0}\biggl(\int_t^\infty s^{q(\alpha/Q-1)}\,ds\biggr)^{1/q}\biggl(\int_0^t ds\biggr)^{1/p'}$$

$$=\bigl((1-\alpha/Q)q-1\bigr)^{-1/q}\sup_{t>0}t^{\alpha/Q-(1/p-1/q)}<\infty\Leftrightarrow 1/p-1/q=\alpha/Q,$$

where

$$C_4\leqq\bigl((1-\alpha/Q)q-1\bigr)^{-1/q}q^{1/q}(q')^{1/p'}=(p')^{1/q}(q')^{1/p'}.$$

Furthermore, from Lemma 4, for the validity of the inequality

$$\biggl(\int_0^\infty\biggl(\int_t^\infty s^{\alpha/Q-1}f_\gamma^*(s)\,ds\biggr)^q dt\biggr)^{1/q}\leqq C_6\biggl(\int_0^\infty f_\gamma^*(t)^p\,dt\biggr)^{1/p}$$

the necessary and sufficient condition is

$$\sup_{t>0}\biggl(\int_0^t ds\biggr)^{1/q}\biggl(\int_t^\infty s^{(\alpha/Q-1)p'}\,ds\biggr)^{1/p'}$$

$$=\bigl((1-\alpha/Q)p'-1\bigr)^{-1/p'}\sup_{t>0}t^{\alpha/Q-(1/p+1/q)}<\infty\Leftrightarrow 1/p-1/q=\alpha/Q,$$

where

$$C_6\leqq\bigl((1-\alpha/Q)p'-1\bigr)^{-1/p'}p^{1/q}(p')^{1/p'}=p^{1/q}q^{1/p'}.$$

By using these inequalities and applying equality (13) we obtain

$$\|K_\alpha\otimes f\|_{L_{q,\gamma}}\leqq C_1(C_4+C_6)\|f\|_{L_{p,\gamma}}.$$

2) Let $p=1$, $f\in L_{1,\gamma}$ and $1-1/q=\alpha/Q$. From (5) and (13) we get

$$\|K_\alpha\otimes f\|_{WL_{q,\gamma}}=\sup_{t>0}t^{1/q}(K_\alpha\otimes f)_\gamma^*(t)$$

$$\leqq C_1\sup_{t>0}t^{1/q}\biggl(t^{\alpha/Q-1}\int_0^t f_\gamma^*(s)\,ds+\int_t^\infty s^{\alpha/Q-1}f_\gamma^*(s)\,ds\biggr)$$

$$=C_1\sup_{t>0}\int_0^t f_\gamma^*(s)\,ds+C_1\sup_{t>0}t^{1/q}\int_t^\infty s^{-1/q}f_\gamma^*(s)\,ds$$

$$\leqq 2C_1\|f_\gamma^*\|_{L_1(0,\infty)}=2C_1\|f\|_{L_{1,\gamma}}.$$

3) Let $p=Q/\alpha$, and $f\in L_{p,1,\gamma}$. By using inequality (5) we have

$$\|K_\alpha\otimes f\|_{L_{\infty,\gamma}}=\sup_{t>0}\,(K_\alpha\otimes f)^*_\gamma(t)$$

$$\leqq C_1\sup_{t>0}\left(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,ds+\int_t^\infty s^{\alpha/Q-1}f^*_\gamma(s)\,ds\right)$$

$$\leqq 2C_1\int_0^\infty s^{-1/p'}f^*_\gamma(s)\,ds=2C_1\|f\|_{L_{p,1,\gamma}}. \quad \square$$

In the following two lemmas we give the $L_{p,\gamma}$-boundedness of $M_{\Omega,\alpha,\gamma}$ and $I_{\Omega,\alpha,\gamma}$ as corollaries of Theorem 3.

COROLLARY 4. *Let* $0<\alpha<Q$, *and* Ω *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega\in L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})$.

1) *If* $1<p<Q/\alpha$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p-1/q=\alpha/Q$, *then* $M_{\Omega,\alpha,\gamma}f$, $I_{\Omega,\alpha,\gamma}f\in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|M_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}\leqq\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}\leqq C_2\,K(p,q)\,\|f\|_{L_{p,\gamma}}.$$

2) *If* $p=1$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1-1/q=\alpha/Q$, *then* $M_{\Omega,\alpha,\gamma}f$, $I_{\Omega,\alpha,\gamma}f\in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|M_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}\leqq\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}\leqq 2\,C_2\|f\|_{L_{p,\gamma}}.$$

3) *If* $p=Q/\alpha$, $f\in L_{p,1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $M_{\Omega,\alpha,\gamma}f$, $I_{\Omega,\alpha,\gamma}f\in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|M_{\Omega,\alpha,\gamma}f\right\|_{L_{\infty,\gamma}}\leqq\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{\infty,\gamma}}\leqq 2\,C_2\|f\|_{L_{p,1,\gamma}}.$$

COROLLARY 5. *Let* $0<\alpha<Q$.

1) *If* $1<p<Q/\alpha$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p-1/q=\alpha/Q$, *then* $M_{\alpha,\gamma}f$, $I_{\alpha,\gamma}f\in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|M_{\alpha,\gamma}f\right\|_{L_{q,\gamma}}\leqq\left\|I_{\alpha,\gamma}f\right\|_{L_{q,\gamma}}\leqq C_3K(p,q)\|f\|_{L_{p,\gamma}}.$$

2) *If* $p=1$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1-1/q=\alpha/Q$, *then* $M_{\alpha,\gamma}f$, $I_{\alpha,\gamma}f\in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|M_{\alpha,\gamma}f\right\|_{WL_{q,\gamma}}\leqq\left\|I_{\alpha,\gamma}f\right\|_{WL_{q,\gamma}}\leqq 2C_3\|f\|_{L_{p,\gamma}}.$$

3) *If* $p = Q/\alpha$, $f \in L_{p,1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $M_{\alpha,\gamma} f$, $I_{\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\big\| M_{\alpha,\gamma} f \big\|_{L_{\infty,\gamma}} \leqq \big\| I_{\alpha,\gamma} f \big\|_{L_{\infty,\gamma}} \leqq 2\, C_3 \|f\|_{L_{p,1,\gamma}}.$$

Note that Corollary 5 was proved in [1] for $k = 1$, and in [5] and [13] for $k = n$ by using other methods, but in these studies the constants were not determined.

5. A sharp rearrangement inequality for the generalized fractional B-maximal operator

The estimates obtained in Corollary 2 are not sharp. We will give a sharp rearrangement estimate for $M_{\Omega,\alpha,\gamma} f$ by using the methods given in [3] in Lemma 7. Our departure point will be the following two estimates involving $\big(M_{\Omega,\alpha,\gamma} f\big)^*_\gamma$:

$$\sup_{t>0} \tau^{1-\alpha/Q} \big(M_{\Omega,\alpha,\gamma} f\big)^*_\gamma(t) \leqq C \int\limits_{\mathbb{R}^n_{k,+}} \big|f(x)\big| (x')^\gamma \, dx, \tag{23}$$

if $\Omega \in L_{Q/(Q-\alpha),\gamma}(\mathbb{S}^{n-1}_{k,+})$, and

$$\sup_{t>0} \big(M_{\Omega,\alpha,\gamma} f\big)^*_\gamma(t) \leqq C \sup_{t>0} t^{\alpha/Q} f^*_\gamma(t), \tag{24}$$

if $\Omega \in L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})$, $s > Q/(Q-\alpha)$, where C depends only on Q and α.

Note that the estimate (23) follows from the second part of Corollary 4. The proof of (24) follows from (15), (22) and Lemma 6. Indeed, for every $r > 0$

$$\frac{1}{r^{Q-\alpha}} \int_{B(0,r)} \big|\Omega(y)\big| T^y \big|f(x)\big| (y')^\gamma \, dy \leqq \frac{r^{Q/s}}{r^{Q-\alpha}} \int_{B(0,r)} \frac{\big|\Omega(y)\big|}{|y|^{Q/s}} T^y \big|f(x)\big| (y')^\gamma \, dy$$

$$\leqq \frac{B}{r^{Q-\alpha}} \int_0^{|B(0,r)|_\gamma} t^{-1/s} f^{**}_\gamma(t) \, dt,$$

where $B = Q^{-1/s} \|\Omega\|_{L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})}$. Note that,

$$f^{**}_\gamma(t) = \frac{1}{t} \sum_{k=1}^\infty \int_{2^{-k}t}^{2^{-k+1}t} f^*_\gamma(s) \, ds \leqq \sum_{k=1}^\infty 2^{-k} \, f^*_\gamma(2^{-k}t)$$

Then

$$\frac{1}{r^{Q-\alpha}}\int_{B(0,r)}\big|\Omega(y)\big|T^y\big|f(x)\big|(y')^\gamma\,dy$$

$$\leqq \frac{B}{r^{Q-\alpha}}\int_0^{|B(0,r)|_\gamma} t^{-1/s}\sum_{k=1}^\infty 2^{-k}\,f_\gamma^*(2^{-k}t)\,dt$$

$$= \frac{B}{r^{Q-\alpha}}\sum_{k=1}^\infty 2^{-k/s}\int_0^{2^{-k}|B(0,r)|_\gamma} t^{-1/s}\,f_\gamma^*(t)\,dt$$

$$\leqq \frac{B}{r^{Q-\alpha}}\sup_{t>0} t^{\alpha/Q} f_\gamma^*(t)\sum_{k=1}^\infty 2^{-k/s}\int_0^{2^{-k}|B(0,r)|_\gamma} t^{-1/s-\alpha/Q}\,dt$$

$$= B_1\sup_{t>0} t^{\alpha/Q} f_\gamma^*(t)\sum_{k=1}^\infty 2^{-k(1-\alpha/Q)} = B_2\sup_{t>0} t^{\alpha/Q} f_\gamma^*(t),$$

and (24) follows, where $B_1=\frac{B\omega(n,k,\gamma)^{1-\alpha/Q}}{(1-\alpha/Q-1/s)}$, and $B_2=\frac{B_1}{2^{1-\alpha/Q}-1}$.

Observe that inequalities (23) and (24) amount to saying that the operator $M_{\Omega,\alpha,\gamma}$ is bounded from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, and from $WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$, respectively.

LEMMA 7. *Let $0<\alpha<Q$, Ω homogeneous of degree zero on $\mathbb{R}^n_{k,+}$, and $\Omega\in L_{s,\gamma}(\mathbb{S}^{n-1}_{k,+})$, $s>Q/(Q-\alpha)$. Then there exists a positive constant C, depending only on n, α and γ such that*

$$(25)\qquad \big(M_{\Omega,\alpha,\gamma}f\big)^*_\gamma(t)\leqq C\sup_{t<\tau<\infty}\tau^{\alpha/Q}f_\gamma^{**}(\tau),\quad t\in(0,\infty),$$

for every $f\in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$.

PROOF. Fix $t\in(0,\infty)$ and let $f\in L^{\mathrm{loc}}_{1,\gamma}$. We may assume that

$$(26)\qquad \sup_{t<\tau<\infty}\tau^{\alpha/Q}f_\gamma^{**}(\tau)<\infty,$$

otherwise (21) holds trivially. Then, by the Hardy–Littlewood inequality (15) (see [7])

$$\int_E\big|f(x)\big|(x')^\gamma\,dx\leqq\int_0^t f_\gamma^*(s)\,ds$$

for every set $E \in \mathbb{R}^n_{k,+}$ with $|E|_\gamma \leqq t$. In particular, if we put $E = \big\{x \in \mathbb{R}^n_{k,+} : \big|f(x)\big| > f^*_\gamma(t)\big\}$, then $|E|_\gamma \leqq t$, and so $f \in L_{1,\gamma}(E)$. Then the function

$$g_t(x) = \max\Big\{\big|f(x)\big| - f^*_\gamma(t), 0\Big\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n_{k,+},$$

belongs to $L_{1,\gamma}$. Also, the function

$$h_t(x) = \min\Big\{\big|f(x)\big|, f^*_\gamma(t)\Big\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n_{k,+},$$

satisfies

$$(h_t)^*_\gamma(\tau) = \min\big\{f^*_\gamma(\tau),\, f^*_\gamma(t)\big\}, \quad \tau \in (0,\infty).$$

Hence,

$$(27) \qquad \sup_{\tau>0} \tau^{\alpha/Q}(h_t)^*_\gamma(\tau) = \max\Big\{\sup_{0<\tau<t} \tau^{\alpha/Q} f^{*(t)}_\gamma,\, \sup_{t\leqq\tau<\infty} \tau^{\alpha/Q} f^*_\gamma(\tau)\Big\}$$

$$= \sup_{t\leqq\tau<\infty} \tau^{\alpha/Q} f^*_\gamma(\tau) \leqq \sup_{t<\tau<\infty} \tau^{\alpha/Q} f^{**}_\gamma(\tau),$$

which, together with (26), implies that $h_t \in WL_{Q/\alpha,\gamma}(\mathbb{R}^n_{k,+})$. Furthermore, since $f = g_t + h_t$, and

$$(28) \qquad (g_t)^*_\gamma(\tau) = \chi_{(0,t)}(\tau)\big(f^*_\gamma(\tau) - f^*_\gamma(t)\big), \quad \tau \in (0,\infty),$$

using (16), (23), (24), (28) and (27), we get

$$\big(M_{\Omega,\alpha,\gamma}f\big)^*_\gamma(t) \leqq \big(M_{\Omega,\alpha,\gamma}g_t\big)^*_\gamma(t/2) + \big(M_{\Omega,\alpha,\gamma}h_t\big)^*_\gamma(t/2)$$

$$\leqq C\bigg((t/2)^{\alpha/Q-1}\int_{\mathbb{R}^n_{k,+}} g_t(y)(y')^\gamma\,dy + \sup_{\tau>0}\tau^{\alpha/Q}(h_t)^*_\gamma(\tau)\bigg)$$

$$\leqq C\bigg(t^{\alpha/Q-1}\int_0^t \big(f^*_\gamma(\tau) - f^*_\gamma(t)\big)\,d\tau + \sup_{t<\tau<\infty}\tau^{\alpha/Q} f^{**}_\gamma(\tau)\bigg)$$

$$\leqq \sup_{t<\tau<\infty}\tau^{\alpha/Q} f^{**}_\gamma(\tau),$$

and the inequality (25) follows. □

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