

(L_p, L_q) boundedness of the fractional maximal operator on the Laguerre hypergroup

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Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator on the Laguerre hypergroup from the spaces $L_p(\mathbb{K})$ to the spaces $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$.

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1. Introduction

In this paper we define the fractional maximal function using harmonic analysis on Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [1,5,7–9]) and we study the fractional maximal function on the Laguerre hypergroup. We obtain the necessary and sufficient conditions for the boundedness of the fractional maximal operator on the Laguerre hypergroup from the spaces $L_p(\mathbb{K})$ to the spaces $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$.

The paper is organized as follows. In Section 2, we give the main result on the boundness of the fractional maximal function on the Laguerre hypergroup. In Section 3, we present some definitions and auxiliary results. In Section 4, we give polar coordinates in Laguerre hypergroup and some lemmas needed to facilitate the proofs of our theorems. The main result of the paper is the boundness of the fractional maximal operator on the Laguerre hypergroup, established in Section 5. We prove the boundedness of the fractional maximal operator from the spaces $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$, $1 < p < (2\alpha + 4)/\beta$, $1/p - 1/q = \beta/(2\alpha + 4)$ and from the spaces $L_1(\mathbb{K})$ to the weak Lebesgue spaces $WL_q(\mathbb{K})$, $1 - 1/q = \beta/(2\alpha + 4)$ and from the spaces $L_{(2\alpha+4)/\beta}(\mathbb{K})$ to $L_\infty(\mathbb{K})$. We show that the conditions on the parameters ensuring that the boundedness cannot be weakened.

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2. Main result

Let $\alpha \geq 0$ be a fixed number, $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and m_α be the weighted Lebesgue measure on \mathbb{K} , given by

$$dm_\alpha(x, t) = \frac{x^{2\alpha+1} dx dt}{\pi \Gamma(\alpha + 1)}, \quad \alpha \geq 0.$$

For every $1 \leq p \leq \infty$, we denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f , measurable on \mathbb{K} such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int_{\mathbb{K}} |f(x, t)|^p dm_\alpha(x, t) \right)^{1/p} < \infty \quad \text{if } p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = \text{ess sup}_{(x,t) \in \mathbb{K}} |f(x, t)| \quad \text{if } p = \infty.$$

For $1 \leq p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions $f(x, t)$, $(x, t) \in \mathbb{K}$ with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r (m_\alpha \{ (x, t) \in \mathbb{K} : |f(x, t)| > r \})^{1/p}.$$

Let $|(x, t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x, t) \in \mathbb{K}$. For $r > 0$ we will denote by $\delta_r(x, t) = (rx, r^2t)$ the dilation of $(x, t) \in \mathbb{K}$, and by $B_r(x, t)$ the ball centered at (x, t) with radius r , i.e., the set of $B_r(x, t) = \{(y, s) \in \mathbb{K} : |(x - y, t - s)|_{\mathbb{K}} < r\}$, and by B_r the ball $B_r(0, 0)$.

We denote by

$$f_r(x, t) = r^{-(2\alpha+4)} f(\delta_{1/r}(x, t))$$

the dilated of the function f defined on \mathbb{K} preserving the mean of f with respect to the measure dm_α , in the sense that

$$\int_{\mathbb{K}} f_r(x, t) dm_\alpha(x, t) = \int_{\mathbb{K}} f(x, t) dm_\alpha(x, t), \quad \forall r > 0 \text{ and } f \in L_1(\mathbb{K}).$$

For $(x, t), (y, s) \in \mathbb{K}$ and $\theta \in [0, 2\pi[, r \in [0, 1]$ let

$$((x, t), (y, s))_{\theta, r} = \left((x^2 + y^2 + 2xyr \cos \theta)^{1/2}, t + s + xy r \sin \theta \right).$$

The generalized translation operators $T_{(x,t)}^{(\alpha)}$ on the Laguerre hypergroup are given for a suitable function f by

$$T_{(x,t)}^{(\alpha)} f(y, s) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(((x, t), (y, s))_{\theta, 1}) d\theta, & \text{if } \alpha = 0, \\ \frac{\alpha}{\pi} \int_0^1 \left(\int_0^{2\pi} f(((x, t), (y, s))_{\theta, r}) d\theta \right) r(1 - r^2)^{\alpha-1} dr, & \text{if } \alpha > 0. \end{cases}$$

Now on the Laguerre hypergroup we define the fractional maximal function by

$$M_\beta f(x, t) = \sup_{r>0} (m_\alpha B_r)^{(\beta/2\alpha+4)-1} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y, s)| dm_\alpha(y, s), \quad 0 \leq \beta < 2\alpha + 4.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator on the Laguerre hypergroup (see [5]).

In ref. [5] the following theorem was proved.

THEOREM 1

(1) *If $f \in L_1(\mathbb{K})$, then $Mf \in WL_1(\mathbb{K})$ and*

$$\|Mf\|_{WL_1(\mathbb{K})} \leq C_1 \|f\|_{L_1(\mathbb{K})},$$

where $C_1 > 0$ is independent of f .

(2) *If $f \in L_p(\mathbb{K})$, $1 < p \leq \infty$, then $Mf \in L_p(\mathbb{K})$ and*

$$\|Mf\|_{L_p(\mathbb{K})} \leq C_2 \|f\|_{L_p(\mathbb{K})},$$

where $C_2 > 0$ is independent of f .

COROLLARY 1 *If $f \in L_{loc}(\mathbb{K})$, then*

$$\lim_{r \rightarrow 0} \frac{1}{m_\alpha B_r} \int_{B_r} |T_{(x,t)}^{(\alpha)} f(y, s) - f(x, t)| dm_\alpha(y, s) = 0$$

for a. e. $(x, t) \in \mathbb{K}$.

For the fractional maximal operator M_β the following theorem is valid.

THEOREM 2 *Let $0 < \beta < 2\alpha + 4$, $1/p - 1/q = \beta/2\alpha + 4$, $1 \leq p \leq 2\alpha + 4/\beta$.*

(1) *If $p = 1$, $f \in L_1(K)$, then for all $\theta > 0$*

$$\int_{\{(x,t) \in K : M_\beta f(x,t) > \theta\}} dm_\alpha(x, t) \leq \left(\frac{C_3}{\theta} \int_K |f(x, t)| dm_\alpha(x, t) \right)^q, \tag{1}$$

where C_3 is independent of f .

(2) *Let $1 < p < 2\alpha + 4/\beta$, $f \in L_p(K)$, then $M_\beta f \in L_q(K)$ and*

$$\left(\int_K (M_\beta f(x, t))^q dm_\alpha(x, t) \right)^{1/q} \leq C_4 \left(\int_K |f(x, t)|^p dm_\alpha(x, t) \right)^{1/p}, \tag{2}$$

where C_4 is independent of f .

(3) *Let $p = 2\alpha + 4/\beta$, $f \in L_p(K)$, then $M_\beta f \in L_\infty(K)$ and*

$$\sup_{(x,t) \in \mathbb{K}} M_\beta f(x, t) \leq C_5 \left(\int_K |f(x, t)|^p dm_\alpha(x, t) \right)^{1/p}, \tag{3}$$

where C_5 is independent of f .

The following theorem is our main result, in which we obtain the necessary and sufficient conditions for the fractional maximal operator M_β to be bounded from the spaces $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$, $1 < p < q < \infty$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$, $1 < q < \infty$.

THEOREM 3 *Let $0 < \beta < 2\alpha + 4$ and $1 \leq p \leq 2\alpha + 4/\beta$.*

- (1) *If $p = 1$, then the condition $1 - 1/q = \beta/2\alpha + 4$ is necessary and sufficient for the boundedness of M_β from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$.*
- (2) *If $1 < p < 2\alpha + 4/\beta$, then the condition $1/p - 1/q = \beta/2\alpha + 4$ is necessary and sufficient for the boundedness of M_β from $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$.*
- (3) *If $p = 2\alpha + 4/\beta$, then M_β is bounded from $L_p(\mathbb{K})$ to $L_\infty(\mathbb{K})$.*

3. Preliminaries

Consider the following partial differential operators system:

$$\begin{cases} D_1 = \frac{\partial}{\partial t}, \\ D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}, \\ (x, t) \in]0, \infty[\times \mathbb{R} \text{ and } \alpha \in [0, \infty[. \end{cases}$$

For $\alpha = n - 1, n \in \mathbb{N} \setminus \{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group \mathbb{H}_n .

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda| \left(m + \frac{\alpha + 1}{2} \right) u; \\ u(0, 0) = 1, \quad \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x, t) = e^{i\lambda t} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2), \quad (x, t) \in \mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on \mathbb{R}_+ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2} L_m^{(\alpha)}(x) / L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [1]).

For $f \in L_1(\mathbb{K})$ the Fourier–Laguerre transform \mathcal{F} is defined by

$$\mathcal{F}(f)(\lambda, m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x, t) f(x, t) dm_{\alpha}(x, t)$$

such that

$$\|\mathcal{F}(f)\|_{L_{\infty}(\mathbb{K})} \leq \|f\|_{L_1(\mathbb{K})}$$

(see [1,8]).

The generalized translation operators $T_{(x,t)}^{(\alpha)}$ on the Laguerre hypergroup satisfy the following properties (see [1,8])

$$\begin{aligned} T_{(x,t)}^{(\alpha)} f(y, s) &= T_{(y,s)}^{(\alpha)} f(x, t), \quad T_{(0,0)}^{(\alpha)} f(y, s) = f(y, s), \\ \|T_{(x,t)}^{(\alpha)} f\|_{L_p(\mathbb{K})} &\leq \|f\|_{L_p(\mathbb{K})} \quad \text{for all } f \in L_p(\mathbb{K}), \quad 1 \leq p \leq \infty, \\ \mathcal{F}(T_{(x,t)}^{(\alpha)} f)(\lambda, m) &= \mathcal{F}(f)(\lambda, m) \varphi_{\lambda,m}(x, t). \end{aligned} \tag{4}$$

The translation operator $T_{(x,t)}^{(\alpha)}$ is defined by

$$T_{(x,t)}^{(\alpha)} f(y, s) = \int_{\mathbb{K}} f(z, v) W_{\alpha}((x, t), (y, s), (z, v)) z^{2\alpha+1} dz dv,$$

where $dzdv$ is the Lebesgue measure on \mathbb{K} , and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha((x, t), (y, s), (z, v)) z^{2\alpha+1} dzdv = 1$$

(see [7]). For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the function $\varphi_{\lambda,m}(x, t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x, t) \varphi_{\lambda,m}(y, s) = T_{(x,t)}^{(\alpha)} \varphi_{\lambda,m}(y, s).$$

By using the generalized translation operators $T_{(x,t)}^{(\alpha)}$, $(x, t) \in \mathbb{K}$, we define a generalized convolution product $*$ on \mathbb{K} by

$$(\delta_{(x,t)} * \delta_{(y,s)})(f) = T_{(x,t)}^{(\alpha)} f(y, s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x, t) .

We define the convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on \mathbb{K} by

$$(\mu * \nu)(f) = \int_{\mathbb{K} \times \mathbb{K}} T_{(x,t)}^{(\alpha)} f(y, s) d\mu(x, t) d\nu(y, s).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with } \check{g}(y, s) = g(y, -s),$$

where, h and g belong to the space $L_1(\mathbb{K})$ of the integrable functions on \mathbb{K} with respect to the measure $dm_\alpha(x, t)$, and $h * g$ is the convolution product defined by

$$(h * g)(x, t) = \int_{\mathbb{K}} T_{(x,t)}^{(\alpha)} h(y, s) g(y, -s) dm_\alpha(y, s), \quad \text{for all } (x, t) \in \mathbb{K}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \leq p, r \leq q \leq \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\mathbb{K})$, and $g \in L_r(\mathbb{K})$, then $f * g \in L_q(\mathbb{K})$ and

$$\|f * g\|_{L_q(\mathbb{K})} \leq \|f\|_{L_p(\mathbb{K})} \|g\|_{L_r(\mathbb{K})}, \tag{5}$$

where $p' = p/(p - 1)$.

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where i is the involution on \mathbb{K} given by $i(x, t) = (x, -t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [2,6]). Hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett and the functions $\varphi_{\lambda,m}$ are characters of \mathbb{K} . If $\alpha = n - 1$ is a non-negative integer, then the Laguerre hypergroup \mathbb{K} can be identified with the hypergroup of radial functions on the Heisenberg group \mathcal{H}_n .

4. Polar coordinates in the Laguerre hypergroup and some lemmas

Let $\Sigma = \Sigma_2$ be the unit sphere in \mathbb{K} . We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [4,5]). For $\xi = (x, t) \in \mathbb{K}$, consider the transformation given by

$$x = r(\cos \varphi)^{1/2}, \quad t = r^2 \sin \varphi,$$

where $-\pi/2 \leq \varphi \leq \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos \varphi)^{1/2}, \sin \varphi) \in \Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos \varphi)^\alpha$. If f is integrable in \mathbb{K} , then

$$\begin{aligned} & \int_{\mathbb{K}} f(x, t) dm_\alpha(x, t) \\ &= \frac{1}{2\pi\Gamma(\alpha + 1)} \int_{-\pi/2}^{\pi/2} \int_0^\infty f(r(\cos \varphi)^{1/2}, r^2 \sin \varphi) r^{2\alpha+3}(\cos \varphi)^\alpha dr d\varphi. \end{aligned}$$

Since

$$\frac{1}{2\pi\Gamma(\alpha + 1)} \int_{-\pi/2}^{\pi/2} (\cos \varphi)^\alpha d\varphi = \int_\Sigma d\xi',$$

we get

$$\int_{\mathbb{K}} f(x, t) dm_\alpha(x, t) = \int_\Sigma \int_0^\infty r^{2\alpha+3} f(\delta_r \xi') dr d\xi'. \tag{6}$$

Here $d\xi'$ is the surface area element on Σ .

LEMMA 1 [4,5] *The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\alpha + 1/2)}{2\sqrt{\pi}\Gamma(\alpha + 1)\Gamma(\alpha/2 + 1)}, \quad \Omega_2 = \frac{\Gamma(\alpha + 1/2)}{4\sqrt{\pi}(\alpha + 2)\Gamma(\alpha + 1)\Gamma(\alpha/2 + 1)}.$$

Note that for any $x \in \mathbb{K}$ and $r > 0$, the area of the sphere $S_r(x, t)$ is $r^{2\alpha+3}\omega_2$ and its volume is $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}(\omega_2/2\alpha + 4)$.

LEMMA 2 [4,5] *The function $f(x, t) = |(x, t)|_{\mathbb{K}}^\lambda$ is integrable in any neighbourhood of the origin if and only if $\lambda > -2\alpha - 4$, and f is integrable in the complement of any neighbourhood of the origin if and only if $\lambda < -2\alpha - 4$.*

LEMMA 3 [5] *For every $(x, t), (y, s) \in \mathbb{K}$ and $r > 0$ the function $T_{(x,t)}^{(\alpha)}\chi_{B_r}(y, s)$ is supported in $\tilde{B}_r(x, t)$ and the following inequality is valid*

$$\begin{aligned} m_\alpha B_r(x, t) &\leq C_5 r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+1}\}, \\ m_\alpha \tilde{B}_r(x, t) &\leq C_6 r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+3}\}, \end{aligned}$$

where $\tilde{B}_r(x, t) = \{(y, s) \in \mathbb{K} : |x - y| < r, |t - s| < x(x + r)\}$.

5. Proof of Theorems 2 and 3

Proof of Theorem 2. The fractional maximal function $M_\beta f(x, t)$ may be interpreted as a fractional maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\mu(E(\xi, 2r)) \leq C_0 \mu(E(\xi, r)) \tag{7}$$

with a constant C_0 independent of ξ and $r > 0$. Here $E(\xi, r) = \{\eta \in X : \rho(\xi, \eta) < r\}$, $\rho(\xi, \eta) = |\xi - \eta|$. Let (X, ρ, μ) be a space of homogeneous type. Define

$$M_{\mu,\beta} f(x) = \sup_{r>0} \mu(E(\xi, r))^{-1+\beta/2\alpha+4} \int_{E(\xi,r)} |f(\eta)| d\mu(\eta), \quad 0 < \beta < 2\alpha + 4.$$

It is well known that the fractional maximal operator $M_{\mu,\beta}$ is of weak type $(1, q)$, $1 - 1/q = \beta/(2\alpha + 4)$ and is bounded from $L_p(X, \mu)$ to $L_q(X, \mu)$, $1/p - 1/q = \beta/(2\alpha + 4)$

for $1 < p < (2\alpha + 4)/\beta$ (see [3]). We shall use this result in the case in which $X = \mathbb{K}$, $\xi = (x, t), \eta = (y, s) \in \mathbb{K}$, $\rho(\xi, \eta) = \max\{|x - y|, (t - s)^2\}$, $d\mu(\xi) = dm_\alpha(x, t)$. It is clear that this measure satisfies the doubling condition (7).

We will show that

$$M_\beta f(x, t) \leq C_7 M_{\mu, \beta} f(x, t). \tag{8}$$

From the definition of the generalized shift operator it follows that $T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s)$ is supported in $\tilde{B}_r(x, t)$.

Moreover, there exists a constant C_8 such that

$$0 \leq T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) \leq \min\{1, C_8 (r/x)^{2\alpha+3}\}, \quad \forall (y, s) \in \tilde{B}_r(x, t). \tag{9}$$

In the case $x \leq r$ this follows from the simple inequality $0 \leq T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) \leq 1$.

Then

$$\begin{aligned} M_\beta f(x, t) &= \sup_{r>0} (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{\mathbb{K}} T_{(x,t)}^{(\alpha)} |f(y, s)| \chi_{B_r}(y, s) dm_\alpha(y, s) \\ &= (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{\mathbb{K}} |f(y, s)| T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) dm_\alpha(y, s) \\ &= (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{\tilde{B}_r(x,t)} |f(y, s)| T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) dm_\alpha(y, s). \end{aligned}$$

Thus

$$M_\beta f(x, t) \leq M_{1,\beta} f(x, t) + M_{2,\beta} f(x, t),$$

where

$$\begin{aligned} M_{1,\beta} f(x, t) &= \sup_{r \geq x} (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y, s)| dm_\alpha(y, s), \\ M_{2,\beta} f(x, t) &= \sup_{r < x} (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y, s)| dm_\alpha(y, s). \end{aligned}$$

If $r \geq x$, then from Lemma 3 $m_\alpha \tilde{B}_r(x, t) \leq C_6 r^{2\alpha+4}$, also $m_\alpha B_r = \Omega_2 r^{2\alpha+4}$ and $T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) \leq 1$ for all $(y, s) \in \tilde{B}_r(x, t)$. Thus yields

$$\begin{aligned} M_{1,\beta} f(x, t) &= \sup_{r \geq x} (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{\tilde{B}_r(x,t)} |f(y, s)| T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) dm_\alpha(y, s) \\ &\leq C_9 \sup_{r>0} (\mu \tilde{B}_r(x, t))^{\beta/(2\alpha+4)-1} \int_{\tilde{B}_r(x,t)} |f(y, s)| d\mu(y, s) = C_9 M_{\mu, \beta} f(x, t). \end{aligned}$$

If $r < x$, then by Lemma 3 and (9) $\mu \tilde{B}_r(x, t) \leq C_6 r x^{2\alpha+3}$ and $T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) \leq C_8 (r/x)^{2\alpha+3}$ for all $(y, s) \in \tilde{B}_r(x, t)$. Thus we have

$$\begin{aligned} M_{2,\beta} f(x, t) &= \sup_{r < x} (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{\tilde{B}_r(x,t)} |f(y, s)| T_{(x,t)}^{(\alpha)} \chi_{B_r}(y, s) dm_\alpha(y, s) \\ &\leq C_{10} \sup_{r>0} (\mu \tilde{B}_r(x, t))^{\beta/(2\alpha+4)-1} \int_{\tilde{B}_r(x,t)} |f(y, s)| d\mu(y, s) = C_{10} M_{\mu, \beta} f(x, t). \end{aligned}$$

Therefore we get (8), which completes the proof of parts (1) and (2).

3) Let $p = 2\alpha + 4/\beta$, $f \in L_p(K)$, then applying the Hölders inequality and the inequality (4) we have

$$\begin{aligned} & (m_\alpha B_r)^{\beta/(2\alpha+4)-1} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y, s)| dm_\alpha(y, s) \\ & \leq (m_\alpha B_r)^{\beta/(2\alpha+4)-1+1/p'} \left(\int_{B_r} \left(T_{(x,t)}^{(\alpha)} |f(y, s)| \right)^p dm_\alpha(y, s) \right)^{1/p} \\ & = \|T_{(x,t)}^{(\alpha)} f\|_{L_p(\mathbb{K})} \leq \|f\|_{L_p(\mathbb{K})} \end{aligned}$$

Theorem 2 has been proved. ■

Proof of Theorem 3. The sufficiency part of the proof follows from Theorem 2.

Necessity. (1) Let M_β bounded from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$.

Define $f_r(x, t) := f(rx, r^2t)$, then

$$\|f_r\|_{L_p(\mathbb{K})} = r^{-2\alpha+4/p} \|f\|_{L_p(\mathbb{K})}$$

and

$$\|M_\beta f_r\|_{WL_q(\mathbb{K})} = r^{-\beta-2\alpha+4/q} \|M_\beta f\|_{WL_q(\mathbb{K})}.$$

By the boundedness M_β from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$

$$\begin{aligned} \|M_\beta f\|_{WL_q(\mathbb{K})} &= r^{\beta+2\alpha+4/q} \|M_\beta f_r\|_{WL_q(\mathbb{K})} \\ &\leq Cr^{\beta+2\alpha+4/q} \|f_r\|_{L_1(\mathbb{K})} = Cr^{\beta+2\alpha+4/q-(2\alpha+4)} \|f\|_{L_1(\mathbb{K})}, \end{aligned}$$

where C depends only on q and α .

If $1 < 1/q + \beta/2\alpha + 4$, then for all $f \in L_1(\mathbb{K})$ we have $\|M_\beta f\|_{WL_q(\mathbb{K})} = 0$ as $r \rightarrow 0$. Similarly, if $1 > 1/q + \beta/2\alpha + 4$, then for all $f \in L_1(\mathbb{K})$ we obtain $\|M_\beta f\|_{WL_q(\mathbb{K})} = 0$ as $r \rightarrow \infty$.

Hence we get $1 = 1/q + \beta/2\alpha + 4$.

(2) Let $1 < p < 2\alpha + 4/\beta$, $f \in L_p(\mathbb{K})$ and assume that the inequality

$$\|M_\beta f\|_{L_q(\mathbb{K})} \leq C \|f\|_{L_p(\mathbb{K})} \tag{10}$$

holds, where C depends only on p, q and β .

We have

$$\|M_\beta f_r\|_{L_q(\mathbb{K})} = r^{-\beta-2\alpha+4/q} \|M_\beta f\|_{L_q(\mathbb{K})}.$$

By the inequality (10)

$$\|M_\beta f\|_{L_q(\mathbb{K})} \leq Cr^{\beta+(2\alpha+4/q)-(2\alpha+4/p)} \|f\|_{L_p(\mathbb{K})}.$$

If $1/p > 1/q + \beta/2\alpha + 4$, then for all $f \in L_p(\mathbb{K})$ we have $\|M_\beta f\|_{L_q(\mathbb{K})} = 0$ as $r \rightarrow 0$, which is impossible. Similarly, if $1/p < 1/q + \beta/2\alpha + 4$, then for all $f \in L_p(\mathbb{K})$ we obtain $\|M_\beta f\|_{L_q(\mathbb{K})} = 0$ as $r \rightarrow \infty$, which is also impossible.

Therefore, $1/p = 1/q + \beta/2\alpha + 4$.

Thus the proof of Theorem 3 is completed. ■

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