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Integral Equations and Operator Theory

O'Neil Inequality for Multilinear Convolutions and Some Applications

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Abstract. In this paper we prove the O'Neil inequality for the k-linear convolution $\mathbf{f} \otimes g$. By using the O'Neil inequality for rearrangements we obtain a pointwise rearrangement estimate of the k-linear convolution. As an application, we obtain necessary and sufficient conditions on the parameters for the boundedness of the k-sublinear fractional maximal operator $M_{\Omega,\alpha}$ and k-linear fractional integral operator $I_{\Omega,\alpha}$ with rough kernels from the spaces $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

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1. Introduction

Multilinear convolution operators and related topics have been research areas of many mathematicians such as R.Coifman and L. Grafakos [4], L. Grafakos [6, 7], L. Grafakos and N. Kalton [8], C.E. Kenig and E.M. Stein [10], Y. Ding and S. Lu [5], V.S. Guliyev and Sh.A. Nazirova [9] and others.

In this paper we consider the k-linear convolution

$$(\mathbf{f} \otimes g)(x) = \int_{\mathbb{R}^n} f_1(x - \theta_1 y) \cdots f_k(x - \theta_k y)\, g(y)\, dy,$$

and prove the O'Neil inequality for this convolution. By using the O'Neil inequality for $\mathbf{f} \otimes g$ we obtain various results for the boundedness of the k-sublinear fractional maximal operator $M_{\Omega,\alpha}$ and k-linear fractional integral $I_{\Omega,\alpha}$ with rough kernels from the spaces operator $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

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Let $k \geq 2$ be an integer and θ_j $(j = 1, 2, \cdots, k)$ be fixed, distinct and nonzero real numbers. We define the k-sublinear fractional maximal function with rough kernel by

$$M_{\Omega,\alpha}(\mathbf{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)| \, |f_1(x-\theta_1 y) \dots f_k(x-\theta_k y)| \, dy,$$

and the k-linear fractional integral with rough kernel by

$$I_{\Omega,\alpha}(\mathbf{f})(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y) \dots f_k(x-\theta_k y) \, dy,$$

where $\Omega \in L_s(S^{n-1})$, $s \geq 1$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and Ω is homogeneous of degree zero on $\mathbb{R}^n$, i.e., $\Omega(tx) = \Omega(x)$ for all $t > 0$, $x \in \mathbb{R}^n$.

This paper organized as follows. In Section 2, we give basic concepts and some lemmas needed to facilitate the proofs of our theorems. In Section 3, we show that the O'Neil inequality for rearrangements of the k-linear convolution holds. In Section 4, we prove the O'Neil inequality for k-linear convolution $\mathbf{f} \otimes g$. In Section 5, we obtain rearrangement estimates for the k-sublinear fractional maximal function $M_{\Omega,\alpha}\mathbf{f}$ and k-linear fractional integral $I_{\Omega,\alpha}\mathbf{f}$ with rough kernels. We prove the boundedness of the operator $M_{\Omega,\alpha}$ and the operator $I_{\Omega,\alpha}$ from the spaces $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $n/(n+\alpha) \leq p < q < \infty$, where p is the harmonic mean of $p_1, p_2, \ldots, p_k$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2. Preliminaries

Let $\mathbb{R}^n$ be the n–dimensional Euclidean space. For the vectors $x = (x_1, ..., x_n)$ and $\xi = (\xi_1, ..., \xi_n)$ in $\mathbb{R}^n$, let $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and denote by $B(x, r)$ the open ball centered at x of radius $r > 0$. It is well known that $|B(x, r)| = v_n r^n$, where $v_n = |B(0, 1)|$ is the volume of the unit ball in $\mathbb{R}^n$.

Let g be a measurable function on $\mathbb{R}^n$. The distribution function of g is defined by the equality

$$\lambda_g(s) = |\{x \in \mathbb{R}^n \ : \ |g(x)| > s\}|, \ s \geq 0.$$

We will denote by $L_0(\mathbb{R}^n)$ the class of all measurable functions g on $\mathbb{R}^n$ which are finite almost everywhere and such that $\lambda_g(s) < \infty$ for all $s > 0$ (see [11]).

If a function g belongs to $L_0(\mathbb{R}^n)$, then its rearrangement is defined by

$$g^*(t) = \inf \{s > 0 \ : \ \lambda_g(s) \leq t\}, \ t \geq 0$$

which is non-increasing on $]0, \infty[$ and equimeasurable with $|g(x)|$, and

$$|\{t > 0 \ : \ g^*(t) > \tau\}| = \lambda_g(\tau)$$

for all $\tau \geq 0$.

Moreover, by the Hardy-Littlewood theorem (see [2], p. 44), for every $f_1, f_2 \in L_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f_1(x)f_2(x)|\,dx \le \int_0^\infty f_1^*(t)f_2^*(t)\,dt.$$

We give some properties of the rearrangements in the following (see, for example [2]):

1) if $0 < t < t+\tau$, then $(g+h)^*(t+\tau) \le g^*(t) + h^*(\tau)$,
2) if $0 < p < \infty$, then

$$\int_{\mathbb{R}^n} |g(x)|^p\,dx = \int_0^\infty (g^*(t))^p\,dt, \tag{2.1}$$

3) for any $t > 0$

$$\sup_{|E|=t} \int_E |g(x)|\,dx = \int_0^t g^*(\tau)\,d\tau.$$

We denote by $WL_p(\mathbb{R}^n)$ the weak L_p space of all measurable functions g with finite norm

$$\|f\|_{WL_p} = \sup_{t>0} t^{1/p} g^*(t) < \infty, \quad 1 \le p < \infty.$$

The function $g^{**} : (0,\infty) \to [0,\infty]$ is defined as $g^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Lemma 2.1. *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$, $k \ge 2$. Then for all $x \in \mathbb{R}^n$ and nonzero real numbers $\theta_1, \dots, \theta_k$,*

$$\int_{\mathbb{R}^n} |f_1(x-\theta_1 y)f_2(x-\theta_2 y)\cdots f_k(x-\theta_k y)|dy \le C_\theta \int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt, \tag{2.2}$$

where $C_\theta = |\theta_1 \dots \theta_k|^{-n}$.

Proof. Applying the Fubini theorem, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} dy \int_0^{|f_1(x-\theta_1 y)|} du_1 \cdots \int_0^{|f_k(x-\theta_k y)|} du_k \\
&\le \int_0^\infty \cdots \int_0^\infty |\{y \ : \ |f_1(\theta_1 y)| > u_1, \dots, |f_k(\theta_k y)| > u_k\}|\,du_1 \cdots du_k \\
&\le C_\theta \int_0^\infty \cdots \int_0^\infty \min\{\lambda_{f_1}(u_1), \dots, \lambda_{f_k}(u_k)\}\,du_1 \cdots du_k \\
&= C_\theta \int_0^\infty \cdots \int_0^\infty |\{t > 0 \ : \ f_1^*(t) > u_1, \dots, f_k^*(t) > u_k\}|\,du_1 \cdots du_k \\
&= C_\theta \int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt.
\end{aligned}$$

Note that, the method of the proof of this theorem was given in [11]. □

We need the following two generalized Hardy inequalities (see [13]) which are to be used in the proof of Theorem 2.

Lemma 2.2. *Let* $1 \le p \le q \le \infty$ *and let* v *and* w *be two functions that are measurable and positive a.e. on* $(0,\infty)$*. Then there exists a constant* C *independent of the function* φ *such that*

$$\left(\int_0^\infty \left(\int_0^t \varphi(\tau)d\tau\right)^q w(t)dt\right)^{1/q} \le C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p}, \tag{2.3}$$

if and only if

$$K = \sup_{t>0}\left(\int_t^\infty w(\tau)d\tau\right)^{1/q}\left(\int_0^t v(\tau)^{1-p'}d\tau\right)^{1/p'} < \infty, \tag{2.4}$$

where $p + p' = pp'$*. Moreover, if* C *is the best constant in* (2.3) *and* K *is defined by* (2.4)*, then*

$$K \le C \le k(p,q)K. \tag{2.5}$$

Here the constant $k(p,q)$ *in* (2.5) *can be written in various forms. For example* (*see* [16])

$$\begin{aligned} & k(p,q) = p^{1/q}(p')^{1/p'} \\ \textit{or}\ \ & k(p,q) = q^{1/q}(q')^{1/p'} \\ \textit{or}\ \ & k(p,q) = \big(1+q/p'\big)^{1/q}\big(1+p'/q\big)^{1/p'}. \end{aligned}$$

Lemma 2.3. *Let* $1 \le p \le q \le \infty$ *and let* v *and* w *be two functions that are measurable and positive a.e. on* $(0,\infty)$*. Then there exists a constant* C *independent of the function* φ *such that*

$$\left(\int_0^\infty \left(\int_t^\infty \varphi(\tau)d\tau\right)^q w(t)dt\right)^{1/q} \le C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p} \tag{2.6}$$

if and only if

$$K_1 = \sup_{t>0}\left(\int_0^t w(\tau)d\tau\right)^{1/q}\left(\int_t^\infty v(\tau)^{1-p'}d\tau\right)^{1/p'} < \infty.$$

Moreover, the best constant C *in* (2.6) *satisfies the inequalities* $K_1 \le C \le k(p,q)K_1$.

Note that Lemmas 2.2, 2.3 was proved by B. Muckenhoupt [14] for $1 \le p = q < \infty$, and by J.S. Bradley [3], V.M. Kokilashvili [12], V.G. Mazya [13] for $p < q$.

In the sequel we shall use the following Lemma, which was proven in [1].

Lemma 2.4. [1] *Let* $0 < p \le 1$, $p \le q < \infty$ *and* k *be a non-negative measurable functions and* u, v *be weight functions on* $(0,\infty)$ *and*

$$T\varphi(x) = \int_0^\infty k(t,\tau)\varphi(\tau)d\tau.$$

Then the inequality

$$\left(\int_0^\infty (T\varphi(t))^q\, u(t)dt\right)^{1/q} \le C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p} \tag{2.7}$$

holds for all non-negative non-increasing functions φ *if and only if*

$$C_0 = \sup_{\rho>0}\left(\int_0^\infty \left(\int_0^\rho k(t,\tau)d\tau\right)^q u(t)dt\right)^{1/q}\left(\int_0^\rho v(t)dt\right)^{-1/p} < \infty.$$

The constant $C = C_0$ *is the best constant in* (2.7).

Corollary 2.5. *Let* $0 < p \le 1$, $p < q < \infty$, $1 < m < \infty$. *Then the inequality*

$$\left(\int_0^\infty \left(\int_t^\infty \tau^{-\frac{1}{m}}\varphi(\tau)\,d\tau\right)^q dt\right)^{1/q} \le \mathbb{C}_0\left(\int_0^\infty (\varphi(t))^p\,dt\right)^{1/p} \tag{2.8}$$

holds for all non-negative non-increasing functions φ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{1}{m'}$, *where*

$$\mathbb{C}_0 = (m')^{1+\frac{1}{q'}}\, B(m', q+1)^{\frac{1}{q}}$$

is the best constant in (2.8), $B(s,r) = \int_0^1 (1-\tau)^{s-1}\tau^{r-1}d\tau$ *is the Beta function.*

Corollary 2.6. *Let* $0 < p < 1$, *then for all non-negative non-increasing functions* φ *the following inequality*

$$\int_0^\infty \left(\frac{1}{t}\int_0^t \varphi(\tau)d\tau\right)^p dt \le p'\int_0^\infty \varphi(t)^p dt \tag{2.9}$$

holds. Here p' *is the best constant in* (2.9).

3. O'Neil inequality for rearrangements of multilinear convolutions

In this section we show that for the multilinear convolution the O'Neil inequality for rearrangements holds. By $\mathbf{f}$ we denote $(f_1, f_2, \cdots, f_k)$ and define

$$\mathbf{f}^*(t) = f_1^*(t)\dots f_k^*(t),$$
$$\mathbf{f}^{**}(t) = \frac{1}{t}\int_0^t f_1^*(\tau)\dots f_k^*(\tau)\,d\tau, \qquad t>0.$$

Lemma 3.1. *Let* $f_1, f_2, \dots, f_k, g \in L_0(\mathbb{R}^n)$. *Then for all* $0 < t < \infty$, *the following inequality holds*

$$(\mathbf{f}\otimes g)^{**}(t) \le C_\theta\left(t\,\mathbf{f}^{**}(t)\; g^{**}(t) + \int_t^\infty \mathbf{f}^*(s)\; g^*(s)\; ds\right). \tag{3.1}$$

Proof. We fix $t > 0$. Let us choose a measurable set E_t in such a way, that

$$\{x\in\mathbb{R}^n \;:\; |g(x)| > g^*(t)\} \subset E_t \subset \{x \in \mathbb{R}^n \;:\; |g(x)| \ge g^*(t)\}.$$

Let

$$g_1(x) = (g(x) - g^*(t))\,\chi_{E_t}(x),\;\; g_2(x) = g(x) - g_1(x).$$

For any measurable set $A \subset \mathbb{R}^n$ with measure $|A| = t$, we have

$$\int_A (\mathbf{f} \otimes g_1)(x)dx = \int_{\mathbb{R}^n} g_1(y)dy \int_A f_1\,(x - \theta_1 y) \cdots f_k\,(x - \theta_k y)\,dx.$$

Hence, taking into account Lemma 2.1, we have

$$\int_A (\mathbf{f} \otimes g_1)(x)dx \le C_\theta \int_0^t f_1^*(u) \cdots f_k^*(u)\;du \int_{\mathbb{R}^n} g_1(y)\;dy$$
$$= C_\theta \int_0^t f_1^*(u) \cdots f_k^*(u)\;du \int_0^t [\;g^*(u) - g^*(t)\;]\;du.$$

Therefore,

$$(\mathbf{f} \otimes g_1)^{**}(t) = \frac{1}{t} \sup_{|A|=t} \int_A (\mathbf{f} \otimes g_1)(x)dx \le C_\theta t\;\mathbf{f}^{**}(t)\;\;[\;g^{**}(t) - g^*(t)\;].$$

By estimating $(\mathbf{f} \otimes g_2)^{**}(t)$ we get

$$(\mathbf{f} \otimes g_2)^{**}(t) = \frac{1}{t} \sup_{|A|=t} \int_A |(\mathbf{f} \otimes g_2)(x)|\,dx.$$

From Lemma 2.1 we have

$$|(\mathbf{f} \otimes g_2)(x)| \le C_\theta \int_0^\infty g_2^*(u) f_1^*(u) f_2^*(u) \cdots f_k^*(u)\;du$$
$$= C_\theta \left(g^*(t) \int_0^t f_1^*(u) f_2^*(u) \cdots f_k^*(u)\;du + \int_t^\infty g^*(u) f_1^*(u) f_2^*(u) \cdots f_k^*(u)\;du \right).$$

Then

$$(\mathbf{f} \otimes g_2)^{**}(t) \le C_\theta \left(t\;\mathbf{f}^{**}(t)\;g^*(t) + \int_t^\infty \mathbf{f}^*(u)\;g^*(u)\;du \right).$$

Consequently, we get (3.1). □

Lemma 3.2. *Let $f_1, f_2, \ldots, f_k, g \in L_0(\mathbb{R}^n)$. Then for any $t > 0$*

$$(\mathbf{f} \otimes g)^{**}(t) \le C_\theta \int_t^\infty \mathbf{f}^{**}(t)\,g^{**}(t)\,dt. \tag{3.2}$$

Proof. We may as well assume the integral on the right of (3.2) is finite and then conclude

$$\lim_{t\to\infty} t\;\mathbf{f}^{**}(t)\;g^{**}(t) = 0. \tag{3.3}$$

By Lemma 3.1 and the fact that $\mathbf{f}^*(t) \le \mathbf{f}^{**}(t)$, we have

$$(\mathbf{f} \otimes g)^{**}(t) \le C_\theta t\;\mathbf{f}^{**}(t)\;g^{**}(t) + C_\theta \int_t^\infty \mathbf{f}^*(u)\;g^*(u)\;du$$
$$\le C_\theta t\;\mathbf{f}^{**}(t)\;g^{**}(t) + C_\theta \int_t^\infty \mathbf{f}^{**}(u)\;g^*(u)\;du. \tag{3.4}$$

Note that since $\mathbf{f}^* = f_1^* \; f_2^* \cdots f_k^*$ and g^* are non-increasing, we have

$$\frac{d}{dt}\;\mathbf{f}^{**}(t) = \frac{1}{t}\,[\;\mathbf{f}^*(t) - \mathbf{f}^{**}(t)\;]$$

and

$$\frac{d}{dt}\;t\;g^{**}(t) = g^*(t)$$

for almost all t. Since $\mathbf{f}^{**}$ and g^{**} are absolutely continuous, we may perform integration by parts and employ (3.3) and (3.4) to obtain

$$\begin{aligned}(\mathbf{f}\otimes g)^{**}(t) &\le C_\theta t\;\mathbf{f}^{**}(t)\;g^{**}(t) + C_\theta u\;\mathbf{f}^{**}(u)\;g^*(u)\;|_t^\infty \\ &\quad + C_\theta \int_t^\infty [\;\mathbf{f}^{**}(u) - \mathbf{f}^*(u)\;]\;g^{**}(u)\;du \\ &= C_\theta \int_t^\infty [\;\mathbf{f}^{**}(u) - \mathbf{f}^*(u)\;]\;g^{**}(u)\;du \\ &\le C_\theta \int_t^\infty \mathbf{f}^{**}(u)\;g^{**}(u)\;du.\end{aligned}$$

□

Theorem 3.3. *Let $f_1, f_2, \ldots, f_k \in L_0(\mathbb{R}^n)$ and $g \in WL_m(\mathbb{R}^n)$, $1 < m < \infty$. Then*

$$\begin{aligned}(\mathbf{f}\otimes g)^*(t) &\le (\mathbf{f}\otimes g)^{**}(t) \\ &\le C_\theta\;\|g\|_{WL_m}\left(m'\;t^{-\frac{1}{m}}\int_0^t \mathbf{f}^*(\tau)\,d\tau + \int_t^\infty \tau^{-\frac{1}{m}}\;\mathbf{f}^*(\tau)\,d\tau\right). \quad (3.5)\end{aligned}$$

Proof. Since $f \in WL_m(\mathbb{R}^n)$, we have

$$g^*(t) \le \|g\|_{WL_m} t^{-\frac{1}{m}}, \quad g^{**}(t) \le m'\|g\|_{WL_m} t^{-\frac{1}{m}}.$$

Taking into account inequality (3.1) we get the inequality (3.5). □

Corollary 3.4. *Let $f_1, f_2, \ldots, f_k \in L_0(\mathbb{R}^n)$ and $g \in WL_m(\mathbb{R}^n)$, $1 < m < \infty$. Then*

$$(\mathbf{f}\otimes g)^*(t) \le (\mathbf{f}\otimes g)^{**}(t) \le m'\;C_\theta\,\|g\|_{WL_m}\int_t^\infty \tau^{-\frac{1}{m}}\;\mathbf{f}^{**}(\tau)\,d\tau. \quad (3.6)$$

4. O'Neil inequality for the multilinear convolutions

In this section we prove the O'Neil inequality for the multilinear convolution $\mathbf{f}\otimes g$. It is said that p is the harmonic mean of $p_1, p_2, \ldots, p_k > 1$, if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$. If $f_j \in L_{p_j}(\mathbb{R}^n)$, for all $j = 1, 2, \cdots, k$, then we say that $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ for $\mathbf{f} = (f_1, f_2, \cdots, f_k)$.

Theorem 4.1 (O'Neil inequality for multilinear convolution). *Suppose that* $1 < m < \infty$, $g \in WL_m(\mathbb{R}^n)$ *and* p *is the harmonic mean of* $p_1, p_2, \dots, p_k > 1$. *If* $m'/(1+m') \le p < m'$ *(equivalently* $1 \le q < \infty$*),* $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *and* q *satisfies* $1/p - 1/q = 1/m'$, *then* $\mathbf{f} \otimes g \in L_q(\mathbb{R}^n)$ *and*

$$\|\mathbf{f} \otimes g\|_q \le C_\theta \, K(p,q,m) \prod_{j=1}^{k} \|f_j\|_{p_j} \|g\|_{WL_m},$$

where

$$K(p,q,m) = \begin{cases} m' \left(\frac{p'}{q}\right)^{\frac{1}{q}} + \left(\frac{q}{p'}\right)^{\frac{1}{p'}}, & \text{if } \ 1 < \mathrm{p} < \mathrm{m}', \\ (m')^{1+\frac{1}{q'}} (p')^{\frac{1}{p}} \ B(m', q+1)^{\frac{1}{q}}, & \text{if } \ \frac{\mathrm{m}'}{1+\mathrm{m}'} \le \mathrm{p} \le 1. \end{cases}$$

Proof. Let $1 < m < \infty$, $\frac{m'}{1+m'} \le p < m'$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{m'}$. Suppose that p is the harmonic mean of $p_1, p_2, \dots, p_k > 1$ and $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$.

Case I. Suppose $1 < p < m'$ (equivalently $m < q < \infty$). By using inequality (3.5) we have

$$\begin{aligned} \|\mathbf{f} \otimes g\|_q &= \left\| (\mathbf{f} \otimes g)^* (t) \right\|_{L_q(0,\infty)} \\ &\le C_\theta \left(\int_0^\infty \left(m' \, t^{-\frac{1}{m}} \int_0^t \mathbf{f}^*(\tau)\, d\tau + \int_t^\infty \tau^{-\frac{1}{m}} \, \mathbf{f}^*(\tau)\, d\tau \right)^q dt \right)^{1/p}. \\ &\le C_\theta \, m' \left(\int_0^\infty \left(\int_0^t \mathbf{f}^*(\tau)\, d\tau \right)^q t^{-\frac{q}{m}} dt \right)^{1/q} \\ &\quad + C_\theta \left(\int_0^\infty \left(\int_t^\infty \tau^{-\frac{1}{m}} \mathbf{f}^*(\tau) d\tau \right)^q dt \right)^{1/q}. \end{aligned}$$

From Lemma 2.2, for the validity of the inequality

$$\left(\int_0^\infty \left(\int_0^t \mathbf{f}^*(\tau)\, d\tau \right)^q t^{-\frac{q}{m}} \, dt \right)^{1/q} \le C_1 \left(\int_0^\infty \left(t^{1/p} \, \mathbf{f}^*(t) \right)^p \frac{dt}{t} \right)^{1/p} \tag{4.1}$$

a necessary and sufficient condition is

$$\begin{aligned} &\sup_{t>0} \left(\int_t^\infty \tau^{-\frac{q}{m}} \, d\tau \right)^{1/q} \left(\int_0^t d\tau \right)^{1/p'} \\ &\qquad = q^{-\frac{1}{q}} \left(\frac{1}{m} - \frac{1}{q} \right)^{-\frac{1}{q}} \sup_{t>0} t^{-\frac{1}{m} + \frac{1}{q} + \frac{1}{p'}} < \infty \Leftrightarrow \frac{1}{p} - \frac{1}{q} = \frac{1}{m'}, \end{aligned}$$

where $p > 1$, $q > m$, $C_1 \le \left(\frac{p'}{q} \right)^{\frac{1}{q}}$.

Furthermore, from Lemma 2.3 for the validity of the inequality

$$\left(\int_0^\infty \left(\int_t^\infty \tau^{-\frac{1}{m}} \mathbf{f}^*(\tau)\, d\tau\right)^q dt\right)^{1/q} \leq C_2 \left(\int_0^\infty \left(t^{1/p}\, \mathbf{f}^*(t)\right)^r \frac{dt}{t}\right)^{1/p} \tag{4.2}$$

a necessary and sufficient condition is

$$\sup_{t>0} \left(\int_0^t d\tau\right)^{1/q} \left(\int_t^\infty \tau^{-\frac{p'}{m}}\, d\tau\right)^{1/p'} = \left(\frac{p'}{m} - 1\right)^{-\frac{1}{p'}} \sup_{t>0} t^{\frac{1}{p'} - \frac{1}{m} + \frac{1}{q}} < \infty \Leftrightarrow \frac{1}{p} - \frac{1}{q} = \frac{1}{m'},$$

where $p < m'$, $C_2 \leq \left(\frac{q}{p'}\right)^{\frac{1}{p'}}$.

By using these inequalities we obtain

$$\begin{aligned}
\|\mathbf{f} \otimes g\, \mathbf{f}\|_q &\leq C_\theta\, (C_1 + C_2) \left(\int_0^\infty \left(t^{1/p}\, \mathbf{f}^*(t)\right)^p \frac{dt}{t}\right)^{1/p} \|g\|_{WL_m} \\
&= C_\theta\, K(p,q,m) \left(\int_0^\infty \prod_{j=1}^k \left(f_j^*(t)\, t^{1/p_j}\right)^p \frac{dt}{t}\right)^{1/p} \|g\|_{WL_m} \\
&\leq C_\theta\, K(p,q,m) \prod_{j=1}^k \left(\int_0^\infty \left(f_j^*(t)\right)^{p_j} dt\right)^{1/p_j} \|g\|_{WL_m} \\
&= C_\theta\, K(p,q,m) \prod_{j=1}^k \|f_j\|_{p_j} \|g\|_{WL_m}.
\end{aligned}$$

Case II. $\frac{m'}{1+m'} \leq p \leq 1$, $p < q < \infty$. From Corollary 2.5, for the validity of the inequality

$$\left(\int_0^\infty \left(\int_t^\infty \tau^{-\frac{1}{m}} \mathbf{f}^{**}(\tau)\, d\tau\right)^q dt\right)^{1/q} \leq \mathbb{C}_0 \left(\int_0^\infty \left(\mathbf{f}^{**}(t)\right)^p dt\right)^{1/p} \tag{4.3}$$

a necessary and sufficient condition is $\frac{1}{p} - \frac{1}{q} = \frac{1}{m'}$. From Corollary 2.6 and Hölder inequality we obtain

$$\begin{aligned}
\|\mathbf{f} \otimes g\|_q &\leq C_\theta\, \mathbb{C}_0\, \|\mathbf{f}^{**}\|_p\, \|g\|_{WL_m} \\
&\leq (p')^{1/p} C_\theta \mathbb{C}_0\, \|\mathbf{f}^*\|_p\, \|g\|_{WL_m} \\
&= (p')^{1/p} C_\theta \mathbb{C}_0\, \|f_1^* \dots f_k^*\|_p\, \|g\|_{WL_m} \\
&= (p')^{1/p} C_\theta \mathbb{C}_0 \prod_{j=1}^k \left\|f_j^*\right\|_{p_j} \|g\|_{WL_m} \\
&= (p')^{1/p} C_\theta \mathbb{C}_0 \prod_{j=1}^k \|f_j\|_{p_j} \|g\|_{WL_m}.
\end{aligned}$$

Thus the proof of Theorem 4.1 is completed. □

5. $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of rough multilinear fractional integral operators

In this section we obtain the Sobolev type theorem for the rough multilinear fractional integral $I_{\Omega,\alpha}\mathbf{f}$.

Lemma 5.1. *Suppose that* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$ *and*

$$g(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}.$$

Then $g \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$ *and*

$$\|g\|_{WL_{n/(n-\alpha)}} = n^{\alpha/n-1} \, \|\Omega\|_{L_{n/(n-\alpha)}(S^{n-1})}. \tag{5.1}$$

Proof. Note that

$$g^*(t) = (nt)^{\alpha/n-1} \, \|\Omega\|_{L_{n/(n-\alpha)}(S^{n-1})}, \quad g^{**}(t) = \frac{n}{\alpha} \, g^*(t),$$

therefore $g \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$ and the equality (5.1) is valid. □

The following Lemma was proved in [5].

Lemma 5.2. [5] *Suppose that* $0 < \alpha < n$, $\Omega \in L_s(S^{n-1})$, $s \geq 1$. *Then*

$$M_{\Omega,\alpha}\mathbf{f}(x) \leq \frac{2^{n-\alpha}}{1 - 2^{\alpha-n}} \, I_{|\Omega|,\alpha}(|\mathbf{f}|)(x), \tag{5.2}$$

where $|\mathbf{f}| = (|f_1|, \ldots, |f_k|)$.

From Lemmas 3.1 and 5.2 we get a pointwise rearrangement estimate of the rough k-sublinear fractional maximal integral $M_{\Omega,\alpha}\mathbf{f}$ and k-linear fractional integral $I_{\Omega,\alpha}\mathbf{f}$ in the following.

Lemma 5.3. *Suppose that* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$ *and* $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$. *Then the following inequalities hold*

$$\begin{aligned}
&(I_{\Omega,\alpha}\mathbf{f})^*(t) \leq (I_{\Omega,\alpha}\mathbf{f})^{**}(t) \\
&\leq C_\theta n^{\frac{\alpha}{n}-1} \, \|\Omega\|_{L_{n/(n-\alpha)}(S^{n-1})} \left(\frac{n}{\alpha} t^{\frac{\alpha}{n}-1} \int_0^t \mathbf{f}^*(\tau)\, d\tau + \int_t^\infty \tau^{\frac{\alpha}{n}-1} \mathbf{f}^*(\tau)\, d\tau \right),
\end{aligned}$$

$$\begin{aligned}
&(M_{\Omega,\alpha}\mathbf{f})^*(t) \leq (M_{\Omega,\alpha}\mathbf{f})^{**}(t) \\
&\leq \frac{2^{n-\alpha} C_\theta}{1 - 2^{\alpha-n}} n^{\frac{\alpha}{n}-1} \, \|\Omega\|_{L_{n/(n-\alpha)}(S^{n-1})} \left(\frac{n}{\alpha} \, t^{\frac{\alpha}{n}-1} \int_0^t \mathbf{f}^*(\tau)\, d\tau + \int_t^\infty \tau^{\frac{\alpha}{n}-1} \mathbf{f}^*(\tau)\, d\tau \right).
\end{aligned}$$

From Theorem 4.1 and Lemma 5.3 we have

Theorem 5.4. [9] *Let* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$, $n/(n+\alpha) \le p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$. *Then* $I_{\Omega,\alpha}\ \mathbf{f} \in L_q(\mathbb{R}^n)$ *and*

$$\|I_{\Omega,\alpha}\ \mathbf{f}\|_q \le C\, \|\Omega\|_{L_{n/(n-\alpha)}(S^{n-1})} \prod_{j=1}^{k} \|f_j\|_{p_j},$$

where $C > 0$ *independent of* $\mathbf{f}$.

Remark 5.5. Note that the cases $\Omega \equiv 1$ and $\Omega \in L_s(S^{n-1})$ for $s > n/(n-\alpha)$ were proven in [6] and [5], respectively.

Finally, in the following theorems we obtain the necessary and sufficient conditions for the rough k-sublinear fractional maximal operator $M_{\Omega,\alpha}$ and k-linear fractional integral operator $I_{\Omega,\alpha}$ to be bounded from the spaces $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1 < p < q < \infty$.

Theorem 5.6. *Let* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$, $n/(n+\alpha) \le p < n/\alpha$. *Then the condition* $1/p - 1/q = \alpha/n$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha}$ *from* $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$.

Proof. Sufficiency part of the theorem follows from Theorem 5.4.
Necessity. Suppose that the operator $I_{\Omega,\alpha}$ is bounded from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, and $n/(n+\alpha) \le p < n/\alpha$ (equivalently $1 \le q < \infty$).

Define $\mathbf{f}_t(x) =: \mathbf{f}(tx)$ for $t > 0$ and $\|\mathbf{f}\|_p =: \prod_{j=1}^{k} \|f_j\|_{p_j}$. Then it can be easily shown that

$$\|\mathbf{f}_t\|_p = \prod_{j=1}^{k} \|(f_j)_t\|_{p_j} = \prod_{j=1}^{k} t^{-\frac{n}{p_j}} \|f_j\|_{p_j} = t^{-\frac{n}{p}} \|\mathbf{f}\|_p,$$

and

$$I_{\Omega,\alpha}\mathbf{f}_t(x) = t^{-\alpha} I_{\Omega,\alpha}\mathbf{f}(tx), \quad \|I_{\Omega,\alpha}\mathbf{f}_t\|_q = t^{-\alpha-\frac{n}{q}} \|I_{\Omega,\alpha}\mathbf{f}\|_q.$$

Since $I_{\Omega,\alpha}\mathbf{f}$ is bounded from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, we have

$$\|I_{\Omega,\alpha}\mathbf{f}\|_q \le C\|\mathbf{f}\|_p,$$

where C is independent of $\mathbf{f}$. Then we get

$$\|I_{\Omega,\alpha}\mathbf{f}\|_q = t^{\alpha+\frac{n}{q}} \|I_{\Omega,\alpha}\mathbf{f}_t\|_q \le Ct^{\alpha+\frac{n}{q}} \|\mathbf{f}_t\|_p = Ct^{\alpha+\frac{n}{q}-\frac{n}{p}} \|\mathbf{f}\|_p.$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n}$, then for all $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ we have $\|I_{\Omega,\alpha}\mathbf{f}\|_q = 0$ as $t \to 0$. If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q}$, then for all $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ we have $\|I_{\Omega,\alpha}\mathbf{f}\|_q = 0$ as $t \to \infty$. Therefore we get $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$ and the proof of the theorem is completed. □

Theorem 5.7. *Let $0 < \alpha < n$, Ω be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$, $n/(n+\alpha) \le p < n/\alpha$. Then the condition $1/p - 1/q = \alpha/n$ is necessary and sufficient for the boundedness of $M_{\Omega,\alpha}$ from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

Proof. Sufficiency part of the theorem follows from the Theorem 5.6 and the inequality (5.2).
Necessity. Suppose that the operator $M_{\Omega,\alpha}$ is bounded from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, and $n/(n+\alpha) \le p < n/\alpha$ (equivalently $1 \le q < \infty$). Then we have

$$M_{\Omega,\alpha}\mathbf{f}_t(x) = t^{-\alpha}\, M_{\Omega,\alpha}\mathbf{f}(tx),$$

and

$$\|M_{\Omega,\alpha}\mathbf{f}_t\|_{L_q(\mathbb{R}^n)} = t^{-\alpha-\frac{n}{q}}\, \|M_{\Omega,\alpha}\mathbf{f}\|_{L_q(\mathbb{R}^n)}\,.$$

By the same argument in Theorem 5.6 we obtain $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$. □

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