

THE SOBOLEV–IL’IN THEOREM FOR THE B -RIESZ POTENTIAL

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Abstract: We study the Riesz potentials that are generated by the generalized shift operator associated to the Laplace–Bessel operator. We obtain an analog of the Sobolev–Il’in theorem for the B -Riesz potential.

Keywords: generalized shift operator, B -Riesz potential, Sobolev–Il’in theorem

Introduction

Sobolev was the first to obtain and substantially use integral representations of a function in terms of its partial derivatives. Similar representations subsequently became one of the principal research tools in embedding theory (see [1, 2]). The Sobolev method of integral representations includes some estimates of potential type integrals as a main ingredient. This important estimates (generalizing the one-dimensional Hardy–Littlewood estimates) were obtained by Sobolev [3] and later augmented by Il’in [4]. More exactly, for the Riesz potential

$$I^\alpha f(x) = (|\cdot|^{-\alpha-N} * f)(x),$$

where $x \in \mathbb{R}^N$ and $0 < \alpha < N$, the well-known Sobolev–Il’in embedding $L_p(\mathbb{R}^N) \hookrightarrow L_q(\mathbb{R}^m)$ was obtained, where $1 < p < q < \infty$ and $1 \leq m \leq N$ with $\frac{N}{p} - \frac{m}{q} = \alpha$.

Using the Fourier–Bessel transform for the Laplace–Bessel operator

$$\Delta_B = \sum_{i=1}^n B_i + \sum_{i=n+1}^N \frac{\partial^2}{\partial x_i^2}, \quad B = (B_1, \dots, B_n), \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i},$$

with $\gamma_i > 0$ for $i = 1, \dots, n$, Kipriyanov and Ivanov [5] (also see [6, 7]) found a fundamental solution in the form of the Riesz potential, the B -Riesz potential. In this article we use the generalized shift operator that is generated by the Laplace–Bessel operator Δ_B to study the B -Riesz potential; namely, we prove for it an analog of the Sobolev–Il’in theorem in $L_{p,\gamma}(\mathbb{R}_{n,+}^N) = L_p(\mathbb{R}_{n,+}^N; x_{1,n}^\gamma dx)$.

Some Notation

Denote by \mathbb{R}^N the N -dimensional Euclidean space of points $x = (x_1, \dots, x_N)$. Put

$$|x| = \left(\sum_{i=1}^N x_i^2 \right)^{1/2}.$$

For $1 \leq n \leq N - 1$, where $N \geq 2$, put $x_{1,n} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $x_{n+1,N} = (x_{n+1}, \dots, x_N) \in \mathbb{R}^{N-n}$; i.e., $x = (x_{1,n}, x_{n+1,N})$ for every point $x \in \mathbb{R}^N$. Put

$$\mathbb{R}_{n,+}^N = \{x \in \mathbb{R}^N : x_1 > 0, \dots, x_n > 0\}.$$

Given $\gamma_1 > 0, \dots, \gamma_n > 0$, put $\gamma \equiv \gamma_{1,n} = (\gamma_1, \dots, \gamma_n)$, $|\gamma| = \gamma_1 + \dots + \gamma_n$, and $x_{1,n}^\gamma = x_1^{\gamma_1} \cdot \dots \cdot x_n^{\gamma_n}$. Given a measurable set E in $\mathbb{R}_{n,+}^N$, put $|E|_\gamma = \int_E x_{1,n}^\gamma dx$.

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Suppose that $1 \leq m \leq N$. In the case $1 \leq m < n \leq N$ we will assume that $x_{1,n} = (x_{1,m}, x_{m+1,n})$, where $x_{m+1,n} = (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m}$, and also that $\gamma = (\gamma_{1,m}, \gamma_{m+1,n})$, where $\gamma_{1,m} = (\gamma_1, \dots, \gamma_m)$ and $\gamma_{m+1,n} = (\gamma_{m+1}, \dots, \gamma_n)$. In the case $1 \leq n < m \leq N$ we will assume that $x_{1,m} = (x_{1,n}, x_{n+1,m})$, where $x_{n+1,m} = (x_{n+1}, \dots, x_m) \in \mathbb{R}^{m-n}$. Given $1 \leq n < m$, define the space $\mathbb{R}_{n,+}^m = \{x_{1,m} \in \mathbb{R}^m : x_1 > 0, \dots, x_n > 0\}$; and, given $m < n$, put

$$\mathbb{R}_{+,+}^m = \{x_{1,m} \in \mathbb{R}^m; x_1 > 0, \dots, x_m > 0\},$$

$$\mathbb{R}_{n-m,+}^{N-m} = \{x_{m+1,N} \in \mathbb{R}^{N-m}; x_{m+1} > 0, \dots, x_n > 0\}.$$

Given a measurable set E in $\mathbb{R}_{n,+}^N$, put $|E|_{\gamma_{1,m}} = \int_E x_{1,m}^{\gamma_{1,m}} dx$.

Denote by $L_{p,\gamma} = L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ the space of all measurable functions $f(x)$ of $x \in \mathbb{R}_{n,+}^N$ of the finite norm

$$\|f\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^N)} = \|f\|_{p,\gamma} = \left(\int_{\mathbb{R}_{n,+}^N} |f(x)|^p x_{1,n}^{\gamma} dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Put $L_{\infty,\gamma}(\mathbb{R}_{n,+}^N) = L_{\infty}(\mathbb{R}_{n,+}^N)$, where $L_{\infty}(\mathbb{R}_{n,+}^N)$ is the set of all essentially bounded functions f of the finite norm

$$\|f\|_{L_{\infty,\gamma}(\mathbb{R}_{n,+}^N)} = \|f\|_{L_{\infty}(\mathbb{R}_{n,+}^N)} = \operatorname{ess\,sup}_{x \in \mathbb{R}_{n,+}^N} |f(x)|.$$

The generalized shift operator (the B -shift operator) is defined as follows (see [5, 7]):

$$T^y f(x) = C_{\gamma} \int_0^{\pi} \cdots \int_0^{\pi} f((x_{1,n}, y_{1,n})_{\alpha}, x_{n+1,N} - y_{n+1,N}) d\nu(\alpha),$$

where

$$C_{\gamma} = \pi^{-\frac{n}{2}} \prod_{i=1}^n \Gamma((\gamma_i + 1)/2) \Gamma^{-1}(\gamma_i/2),$$

$$(x_{1,n}, y_{1,n})_{\alpha} = ((x_1, y_1)_{\alpha_1}, \dots, (x_n, y_n)_{\alpha_n}),$$

$$(x_i, y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2}, \quad 1 \leq i \leq n,$$

$$d\nu(\alpha) = \prod_{i=1}^n \sin^{\gamma_i-1} \alpha_i d\alpha_1 \dots d\alpha_n, \quad 1 \leq n \leq N.$$

Introduce the following additional notation: in case $1 \leq n < m \leq N$,

$$(x_{1,m}, y_{1,m})_{\alpha_{1,m}} = ((x_{1,n}, y_{1,n})_{\alpha}, x_{n+1,m} - y_{n+1,m}),$$

$$T_{x_{1,m}}^{y_{1,m}} f(x) = C_{\gamma} \int_0^{\pi} \cdots \int_0^{\pi} f((x_{1,n}, y_{1,n})_{\alpha}, x_{n+1,m} - y_{n+1,m}, x_{m+1,n}) d\nu(\alpha),$$

and in case $1 \leq m < n \leq N$,

$$(x_{1,m}, y_{1,m})_{\alpha_{1,m}} = ((x_1, y_1)_{\alpha_1}, \dots, (x_m, y_m)_{\alpha_m}),$$

$$(x_{m+1,n}, y_{m+1,n})_{\alpha_{m+1,n}} = ((x_{m+1}, y_{m+1})_{\alpha_{m+1}}, \dots, (x_n, y_n)_{\alpha_n}),$$

$$T_{x_{1,m}}^{y_{1,m}} f(x) = C_{\gamma_{1,m}} \int_0^{\pi} \cdots \int_0^{\pi} f((x_{1,m}, y_{1,m})_{\alpha_{1,m}}, x_{m+1,n}) d\nu(\alpha_{1,m}),$$

$$T_{x_{m+1},N}^{y_{m+1},N} f(x) = C_{\gamma_{m+1},n} \int_0^\pi \cdots \int_0^\pi f(x_{1,m}, \\ (x_{m+1,n}, y_{m+1,n})_{\alpha_{m+1,n}}, x_{n+1,N} - y_{n+1,N}) d\nu(\alpha_{m+1,n}),$$

where

$$C_{\gamma_{1,m}} = \pi^{-\frac{m}{2}} \prod_{i=1}^m \Gamma((\gamma_i + 1)/2) \Gamma^{-1}(\gamma_i/2), \\ d\nu(\alpha_{1,m}) = \prod_{i=1}^m \sin^{\gamma_i-1} \alpha_i d\alpha_1 \cdots d\alpha_m, \\ C_{\gamma_{m+1},n} = \pi^{-\frac{n-m}{2}} \prod_{i=m+1}^n \Gamma((\gamma_i + 1)/2) \Gamma^{-1}(\gamma_i/2), \\ d\nu(\alpha_{m+1,n}) = \prod_{i=m+1}^n \sin^{\gamma_i-1} \alpha_i d\alpha_{m+1} \cdots d\alpha_n.$$

Observe that in case $1 \leq m < n \leq N$

$$T^y f(x) = T_{x_{1,m}}^{y_{1,m}} T_{x_{m+1},N}^{y_{m+1},N} f(x), \quad C_\gamma = C_{\gamma_{1,m}} C_{\gamma_{m+1},n}.$$

The Main Results

For the B -Riesz potential

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} f(y) y_{1,n}^\gamma dy, \quad 0 < \alpha < N + |\gamma|,$$

the following Sobolev theorem holds:

Theorem 1. *Suppose that $0 < \alpha < N + |\gamma|$.*

- (a) *If $1 \leq p < \frac{N+|\gamma|}{\alpha}$ and $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ then $I_\gamma^\alpha f$ converges absolutely for almost all $x \in \mathbb{R}_{n,+}^N$.*
- (b) *If $1 < p < \frac{N+|\gamma|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{N+|\gamma|}$, and $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ then $I_\gamma^\alpha f \in L_{q,\gamma}(\mathbb{R}_{n,+}^N)$ and*

$$\|I_\gamma^\alpha f\|_{q,\gamma} \leq C \|f\|_{p,\gamma}$$

with some constant $C > 0$ independent of f .

- (c) *If $f \in L_{1,\gamma}(\mathbb{R}_{n,+}^N)$ and $1/q = 1 - \frac{\alpha}{N+|\gamma|}$, then*

$$|\{x \in \mathbb{R}_{n,+}^N : I_\gamma^\alpha f(x) > \beta\}|_\gamma^{1/q} \leq \frac{C}{\beta} \|f\|_{1,\gamma}$$

with some constant $C > 0$ independent of f .

REMARK 1. Theorem 1 is proved by different methods in [8, 9]; also see [10]. In the case $n = 1$ Theorem 1 is proved in [11], and in the case $n = N$, in [12]; also see [13].

REMARK 2. Examples demonstrate (see, for instance, [13, p. 323]) that if $p \geq \frac{N+|\gamma|}{\alpha}$ then the B -Riesz potential $I_\gamma^\alpha f$ need not be defined for all $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$.

In this article we prove the following two main theorems analogous to the Sobolev–Il'in theorem for the B -Riesz potential.

Theorem 2. Suppose that $0 < \alpha < N + |\gamma|$, $1 \leq p < \frac{N+|\gamma|}{\alpha}$, and $1 \leq m < n \leq N$.

(a) If $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ then for every $x_{m+1,N} \in \mathbb{R}_{n-m,+}^{N-m}$ and almost all $x_{1,m} \in \mathbb{R}_{+,+}^m$ the B-Riesz potential $I_\gamma^\alpha f(x)$ exists.

(b) If $1 < p < \frac{N+|\gamma|}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$, and $\alpha = \frac{N+|\gamma|}{p} - \frac{m+|\gamma_{1,m}|}{q}$ then

$$I_\gamma^\alpha f(\cdot, x_{m+1,N}) \in L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)$$

for every $x_{m+1,N} \in \mathbb{R}_{n-m,+}^{N-m}$ and

$$\|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} \leq C \|f\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^N)} \quad (1)$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

(c) If $f \in L_{1,\gamma}(\mathbb{R}_{n,+}^N)$ and $\alpha = N + |\gamma| - \frac{m+|\gamma_{1,m}|}{q}$ then for every $x_{m+1,N} \in \mathbb{R}_{n-m,+}^{N-m}$

$$\{x_{1,m} \in \mathbb{R}_{+,+}^m : |I_\gamma^\alpha f(x_{1,m}, x_{m+1,N})| > \beta\}_{\gamma_{1,m}}^{1/q} \leq \frac{C}{\beta} \|f\|_{L_{1,\gamma}(\mathbb{R}_{n,+}^N)} \quad (2)$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

Theorem 3. Suppose that $0 < \alpha < N + |\gamma|$, $1 \leq p < \frac{N+|\gamma|}{\alpha}$ and $1 \leq n \leq m \leq N$.

(a) If $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ then for every $x_{m+1,N} \in \mathbb{R}^{N-m}$ and almost all $x_{1,m} \in \mathbb{R}_{n,+}^m$ the B-Riesz potential $I_\gamma^\alpha f(x)$ exists.

(b) If $1 < p < \frac{N+|\gamma|}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}_{n,+}^N)$ and $\frac{N+|\gamma|}{p} - \frac{m+|\gamma|}{q} = \alpha$ then $I_\gamma^\alpha f(\cdot, x_{m+1,N}) \in L_{q,\gamma}(\mathbb{R}_{n,+}^m)$ for every $x_{m+1,N} \in \mathbb{R}^{N-m}$ and

$$\|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)} \leq C \|f\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^N)} \quad (3)$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

(c) If $f \in L_{1,\gamma}(\mathbb{R}_{n,+}^N)$ and $\alpha = N + |\gamma| - \frac{m+|\gamma|}{q}$ then for every $x_{m+1,N} \in \mathbb{R}^{N-m}$

$$\{x_{1,m} \in \mathbb{R}_{n,+}^m : |I_\gamma^\alpha f(x_{1,m}, x_{m+1,N})| > \beta\}_\gamma^{1/q} \leq \frac{C}{\beta} \|f\|_{L_{1,\gamma}(\mathbb{R}_{n,+}^N)} \quad (4)$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

REMARK 3. For the anisotropic B-Riesz potential an analog of (3) for $n = 1$ and $1 < m \leq N$ is obtained in [14].

Auxiliary Lemmas

The proofs of Theorems 2 and 3 use the following two auxiliary lemmas, which we prove right here.

Lemma 1. Suppose that $1 \leq p < \infty$ and $1 \leq m < n \leq N$. Suppose further that $x = (x_{1,m}, x_{m+1,N})$ with $x_{1,m} \in \mathbb{R}_{+,+}^m$ and $x_{m+1,N} \in \mathbb{R}_{n-m,+}^{N-m}$, and that $0 < \alpha < \frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'}$ (while $0 < \alpha < N + |\gamma|$ in the case $p = 1$). Then there exists a constant C_1 depending on p, m, n, α, γ such that for all $f \in L_{1,\gamma}^{\text{loc}}(\mathbb{R}_{n,+}^N)$ and $x \in \mathbb{R}_{n,+}^N$ we have

$$\begin{aligned} & \int_{\mathbb{R}_{n,+}^N} T y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ & \leq C_1 \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha - (\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'})} \|f(y_{1,m}, \cdot)\|_{L_{p,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m}, \end{aligned} \quad (5)$$

where

$$C_1 = \|(1 + |x_{m+1,n}|^2)^{\frac{\alpha-N-|\gamma|}{2}}\|_{L_{p',\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})}$$

and $p' = \frac{p}{p-1}$ in the case $p > 1$, and $C_1 = C_{\gamma_{m+1,n}}$ in the case $p = 1$.

PROOF. Consider the case $p = 1$. We have

$$\begin{aligned} T^y|x|^{\alpha-N-|\gamma|} &= C_\gamma \int_0^\pi \cdots \int_0^\pi |((x_{1,n}, y_{1,n})_\alpha, x_{n+1,N} - y_{n+1,N})|^{\alpha-N-|\gamma|} d\nu(\alpha) \\ &\leq C_\gamma \int_0^\pi \cdots \int_0^\pi |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{\alpha-N-|\gamma|} d\nu(\alpha_{1,m}) = C_{\gamma_{m+1,n}} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|}. \end{aligned}$$

The last inequality yields

$$\begin{aligned} \int_{\mathbb{R}_{n,+}^N} T^y|x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy &\leq C_{\gamma_{m+1,n}} \int_{\mathbb{R}_{n,+}^N} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ &= C_{\gamma_{m+1,n}} \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} \|f(y_{1,m}, \cdot)\|_{L_{1,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m}. \end{aligned}$$

Consequently, (5) holds for $p = 1$ with $C_1 = C_{\gamma_{m+1,n}}$.

Suppose now that $p > 1$. Then

$$\int_{\mathbb{R}_{n,+}^N} T^y|x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy = \int_{\mathbb{R}_{+,+}^m} F(x, y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m},$$

where

$$F(x, y_{1,m}) = \int_{\mathbb{R}_{n-m,+}^{N-m}} T^y|x|^{\alpha-N-|\gamma|} |f(y)| y_{m+1,n}^{\gamma_{m+1,n}} dy_{m+1,N}.$$

The finiteness of the right-hand side of (5) implies that $\|f(y_{1,m}, \cdot)\|_{L_{p,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} < \infty$ for almost all $y_{1,m} \in \mathbb{R}_{+,+}^m$. Applying the Hölder inequality for $F(x, y_{1,m})$, we obtain

$$F(x, y_{1,m}) \leq \|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p',\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} \|f(y_{1,m}, \cdot)\|_{L_{p,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})}.$$

In order to estimate $\|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p',\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})}$, observe that

$$\begin{aligned} T^y|x|^{\alpha-N-|\gamma|} &= T_{x_{1,m}}^{y_{1,m}} T_{x_{m+1,N}}^{y_{m+1,N}} (|x_{1,m}|^2 + |x_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}} \\ &= C_{\gamma_{1,m}} \int_0^\pi \cdots \int_0^\pi T_{x_{m+1,N}}^{y_{m+1,N}} (|(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^2 + |x_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}} d\nu(\alpha_{1,m}). \end{aligned}$$

Using the generalized Minkowski inequality, the Hölder inequality, and the properties of the generalized shift operator, we arrive at

$$\|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p',\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} \leq C_{\gamma_{1,m}} \int_0^\pi \cdots \int_0^\pi J(x_{1,m}, y_{1,m}, \alpha_{1,m}) d\nu(\alpha_{1,m}),$$

where

$$\begin{aligned}
J(x_{1,m}, y_{1,m}, \alpha_{1,m}) &= \left[\int_{\mathbb{R}_{n-m,+}^{N-m}} (|(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^2 + |y_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}p'} y_{m+1,n}^{\gamma_{m+1,n}} dy_{m+1,N} \right]^{1/p'} \\
&= |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{\alpha-N-|\gamma|} \left[\int_{\mathbb{R}_{n-m,+}^{N-m}} \left(1 + \left(\frac{|y_{m+1,N}|}{|(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|} \right)^2 \right)^{\frac{\alpha-N-|\gamma|}{2}p'} y_{m+1,n}^{\gamma_{m+1,n}} dy_{m+1,N} \right]^{1/p'}.
\end{aligned}$$

Making the change of variables $\frac{y_{m+1,N}}{|(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|} = z_{m+1,N}$ and considering that

$$\begin{aligned}
y_{m+1,N} &= |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}| z_{m+1,N}, \\
dy_{m+1,N} &= |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{N-m} dz_{m+1,N}, \\
y_{m+1,n}^{\gamma_{m+1,n}} dy_{m+1,N} &= |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{N-m+|\gamma_{m+1,n}|} z_{m+1,n}^{\gamma_{m+1,n}} dz_{m+1,N},
\end{aligned}$$

we obtain

$$J(x_{1,m}, y_{1,m}, \alpha_{1,m}) = C_1 |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{\alpha-N-|\gamma| + \frac{N-m+|\gamma_{m+1,n}|}{p'}},$$

where

$$\begin{aligned}
C_1 &= \left[\int_{\mathbb{R}_{n-m,+}^{N-m}} (1 + |z_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}p'} z_{m+1,n}^{\gamma_{m+1,n}} dz_{m+1,N} \right]^{1/p'} \\
&= \|(1 + |z_{m+1,n}|^2)^{\frac{\alpha-N-|\gamma|}{2}}\|_{L_{p', \gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} < \infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|T^y |x|^{\alpha-N-|\gamma|}\|_{L_{p', \gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} \\
&\leq C_1 C_{\gamma_{1,m}} \int_0^\pi \cdots \int_0^\pi |(x_{1,m}, y_{1,m})_{\alpha_{1,m}}|^{\alpha-N-|\gamma| + \frac{N-m+|\gamma_{m+1,n}|}{p'}} d\nu(\alpha_{1,m}) \\
&= C_1 T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma| + \frac{N-m+|\gamma_{m+1,n}|}{p'}}.
\end{aligned}$$

Since

$$\begin{aligned}
\alpha - N - |\gamma| + \frac{N-m+|\gamma_{m+1,n}|}{p'} &= \alpha - \left(N + |\gamma| - \frac{N-m+|\gamma_{m+1,n}|}{p'} \right) \\
&= \alpha - \left(\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'} \right),
\end{aligned}$$

it follows that

$$\|T^y |x|^{\alpha-N-|\gamma|}\|_{L_{p', \gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} \leq C_1 T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha - \left(\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'} \right)}.$$

Having these estimates, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\
&\leq C_1 \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha - \left(\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'} \right)} \|f(y_{1,m}, \cdot)\|_{L_{p, \gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m}.
\end{aligned}$$

Consequently, (5) holds for $p > 1$ with some constant $C_1 > 0$.

The proof of Lemma 1 is complete.

Lemma 2. Suppose that $1 \leq p < \infty$ and $1 \leq n \leq m \leq N$. Suppose further that $x = (x_{1,m}, x_{m+1,N})$ with $x_{1,m} \in \mathbb{R}_{n,+}^m$ and $x_{m+1,N} \in \mathbb{R}^{N-m}$, and that $0 < \alpha < |\gamma| + \frac{N}{p} + \frac{m}{p'}$ (while $0 < \alpha < N + |\gamma|$ in the case $p = 1$). Then there exists a constant C_2 depending on p, m, n, α, γ such that for every $f \in L_{1,\gamma}^{\text{loc}}(\mathbb{R}_{n,+}^N)$ and all $x \in \mathbb{R}_{n,+}^N$ we have

$$\begin{aligned} & \int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ & \leq C_2 \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-|\gamma|-(\frac{N}{p}+\frac{m}{p'})} \|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m}, \end{aligned} \quad (6)$$

where

$$C_2 = \|(1 + |x_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}}\|_{L_{p'}(\mathbb{R}^{N-m})}$$

and $p' = \frac{p}{p-1}$ in the case $p > 1$ and $C_2 = 1$ in the case $p = 1$.

PROOF. Consider the case $p = 1$. We have

$$\begin{aligned} T^y |x|^{\alpha-N-|\gamma|} &= C_\gamma \int_0^\pi \dots \int_0^\pi |((x_{1,n}, y_{1,n})_\alpha, x_{n+1,m} - y_{n+1,m}, x_{m+1,N} - y_{m+1,N})| d\nu(\alpha) \\ &\leq C_\gamma \int_0^\pi \dots \int_0^\pi |((x_{1,n}, y_{1,n})_\alpha, x_{n+1,m} - y_{n+1,m})| d\nu(\alpha) = T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|}. \end{aligned}$$

The last inequality implies that

$$\begin{aligned} & \int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \leq \int_{\mathbb{R}_{n,+}^N} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ &= \int_{\mathbb{R}_{n,+}^m} \int_{\mathbb{R}^{N-m}} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy_{1,m} dy_{m+1,N} \\ &= \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} \|f(y_{1,m}, \cdot)\|_{L_1(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m}. \end{aligned}$$

Consequently, (6) holds for $p = 1$ with the constant $C_2 = 1$.

Suppose now that $p > 1$. Then

$$\int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy = \int_{\mathbb{R}_{n,+}^m} F(x, y_{1,m}) y_{1,n}^\gamma dy_{1,m},$$

where

$$F(x, y_{1,m}) = \int_{\mathbb{R}^{N-m}} T^y |x|^{\alpha-N-|\gamma|} |f(y)| dy_{m+1,N}.$$

The finiteness of the right-hand side of (6) implies that for almost all $y_{1,m} \in \mathbb{R}_{n,+}^m$

$$\|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})} < \infty.$$

Using the Hölder inequality for $F(x, y_{1,m})$ we arrive at

$$F(x, y_{1,m}) \leq \|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p'}(\mathbb{R}^{N-m})} \|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})}.$$

In order to estimate $\|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p'}(\mathbb{R}^{N-m})}$, observe that

$$T^y|x|^{\alpha-N-|\gamma|} = C_\gamma \int_0^\pi \cdots \int_0^\pi \left[|(x_{1,m}, y_{1,m})_\alpha|^2 + \sum_{i=m+1}^N |x_i - y_i|^2 \right]^{\frac{\alpha-N-|\gamma|}{2}} d\nu(\alpha).$$

Applying the generalized Minkowski inequality and

$$\begin{aligned} & \int_{\mathbb{R}^{N-m}} (|(x_{1,m}, y_{1,m})_\alpha|^2 + |x_{m+1,N} - y_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2} p'} dy_{m+1,N} \\ &= \int_{\mathbb{R}^{N-m}} (|(x_{1,m}, y_{1,m})_\alpha|^2 + |y_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2} p'} dy_{m+1,N}, \end{aligned}$$

we obtain

$$\|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p'}(\mathbb{R}^{N-m})} \leq C_\gamma \int_0^\pi \cdots \int_0^\pi J(x_{1,m}, y_{1,m}, \alpha) d\nu(\alpha),$$

where

$$\begin{aligned} J(x_{1,m}, y_{1,m}, \alpha) &= \left[\int_{\mathbb{R}^{N-m}} (|(x_{1,m}, y_{1,m})_\alpha|^2 + |y_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2} p'} dy_{m+1,N} \right]^{1/p'} \\ &= |(x_{1,m}, y_{1,m})_\alpha|^{\alpha-N-|\gamma|} \left[\int_{\mathbb{R}^{N-m}} \left(1 + \left(\frac{|y_{m+1,N}|}{|(x_{1,m}, y_{1,m})_\alpha|} \right)^2 \right)^{\frac{\alpha-N-|\gamma|}{2} p'} dy_{m+1,N} \right]^{1/p'}. \end{aligned}$$

Making the change of variables $\frac{y_{m+1,N}}{|(x_{1,m}, y_{1,m})_\alpha|} = z_{m+1,N}$ and considering that

$$y_{m+1,N} = |(x_{1,m}, y_{1,m})_\alpha| z_{m+1,N}, \quad dy_{m+1,N} = |(x_{1,m}, y_{1,m})_\alpha|^{N-m} dz_{m+1,N},$$

we obtain

$$J(x_{1,m}, y_{1,m}, \alpha) = C_2 |(x_{1,m}, y_{1,m})_\alpha|^{\alpha-N-|\gamma| + \frac{N-m}{p'}},$$

where

$$C_2 = \left[\int_{\mathbb{R}^{N-m}} (1 + |z_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2} p'} dz_{m+1,N} \right]^{1/p'} = \|(1 + |z_{m+1,N}|^2)^{\frac{\alpha-N-|\gamma|}{2}}\|_{L_{p'}(\mathbb{R}^{N-m})} < \infty.$$

Consequently,

$$\begin{aligned} & \|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p'}(\mathbb{R}^{N-m})} \\ & \leq C_2 C_\gamma \int_0^\pi \cdots \int_0^\pi |(x_{1,m}, y_{1,m})_\alpha|^{\alpha-N-|\gamma| + \frac{N-m}{p'}} d\nu(\alpha) = C_2 T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma| + \frac{N-m}{p'}}. \end{aligned}$$

Since $\alpha - N - |\gamma| + \frac{N-m}{p'} = \alpha - |\gamma| - \left(\frac{N}{p} + \frac{m}{p'} \right)$, it follows that

$$\|T^y|x|^{\alpha-N-|\gamma|}\|_{L_{p'}(\mathbb{R}^{N-m})} \leq C_2 T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-|\gamma| - \left(\frac{N}{p} + \frac{m}{p'} \right)}.$$

These estimates yield

$$\int_{\mathbb{R}_{n,+}^N} T^y|x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \leq C_2 \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-|\gamma| - \left(\frac{N}{p} + \frac{m}{p'} \right)} \|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m}.$$

Consequently, (6) holds for $p > 1$ with some constant $C_2 > 0$.

The proof of Lemma 2 is complete.

Proofs of Theorems 2 and 3

PROOF OF THEOREM 2. Claim (a) of Theorem 2 follows directly from Lemma 1 and Theorem 1. Let us prove claims (b) and (c) of Theorem 2.

Consider the case $p = 1$. Lemma 1 and Theorem 1 yield

$$\begin{aligned} & \left| \left\{ x_{1,m} \in \mathbb{R}_{+,+}^m : |I_\gamma^\alpha f(x_{1,m}, x_{m+1,N})| > \beta \right\} \right|_{\gamma_{1,m}}^{1/q} \\ & \leq \left| \left\{ x_{1,m} \in \mathbb{R}_{+,+}^m : \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} \|f(y_{1,m}, \cdot)\|_{L_{1,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m} > \frac{\beta}{C_1} \right\} \right|_{\gamma_{1,m}}^{1/q} \\ & \leq \frac{C_1}{\beta} \int_{\mathbb{R}_{+,+}^m} \|f(y_{1,m}, \cdot)\|_{L_{1,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m} = \frac{C_1}{\beta} \|f\|_{L_{1,\gamma}(\mathbb{R}_{n,+}^N)}. \end{aligned}$$

Turn now to the case $1 < p < q < \infty$. Using Lemma 1 and making the change of variables

$$h(y_{1,m}) = \|f(y_{1,m}, \cdot)\|_{L_{p,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})},$$

we obtain

$$\begin{aligned} |I_\gamma^\alpha f(x)| & \leq \int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ & \leq C_1 \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha - (\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'})} \|f(y_{1,m}, \cdot)\|_{L_{p,\gamma_{m+1,n}}(\mathbb{R}_{n-m,+}^{N-m})} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \\ & = C_1 \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha - (\frac{N+|\gamma|}{p} + \frac{m+|\gamma_{1,m}|}{p'})} h(y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m}. \end{aligned}$$

Since $\alpha - \frac{N+|\gamma|}{p} - \frac{m+|\gamma_{1,m}|}{p'} = -(m+|\gamma_{1,m}|)(\frac{1}{p'} + \frac{1}{q})$, it follows that

$$\|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} \leq C_1 \left\| \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma_{1,m}|)(\frac{1}{p'} + \frac{1}{q})} h(y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)}.$$

Since

$$\begin{aligned} & \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma_{1,m}|)(\frac{1}{p'} + \frac{1}{q})} h(y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m} \\ & = \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\beta - m - |\gamma_{1,m}|} h(y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m} = I_{\gamma_{1,m}}^\beta h(x_{1,m}), \end{aligned}$$

where $\beta = \frac{m+|\gamma_{1,m}|}{p} - \frac{m+|\gamma_{1,m}|}{q} > 0$ and $N = n = m$, applying Theorem 1 for $N = n = m$ and considering that $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{m+|\gamma_{1,m}|}$, we obtain

$$\begin{aligned} \|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} & = \left\| \int_{\mathbb{R}_{+,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma_{1,m}|)(\frac{1}{p'} + \frac{1}{q})} h(y_{1,m}) y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} \\ & = \|I_{\gamma_{1,m}}^\beta h\|_{L_{q,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} \leq C \|h\|_{L_{p,\gamma_{1,m}}(\mathbb{R}_{+,+}^m)} = C \|f\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^N)} \end{aligned}$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

The proof of Theorem 2 is complete.

PROOF OF THEOREM 3. Claim (a) of Theorem 3 is immediate from Lemma 2 and Theorem 1. Let us prove claims (b) and (c) of Theorem 3.

Consider the case $p = 1$. Lemma 2 and Theorem 1 yield

$$\begin{aligned} & \left| \{x_{1,m} \in \mathbb{R}_{n,+}^m : |I_\gamma^\alpha f(x_{1,m}, x_{m+1,N})| > \beta\} \right|_\gamma^{1/q} \\ & \leq \left| \left\{ x_{1,m} \in \mathbb{R}_{n,+}^m : \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-N-|\gamma|} \|f(y_{1,m}, \cdot)\|_{L_1(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m} > \frac{\beta}{C_2} \right\} \right|_\gamma^{1/q} \\ & \leq \frac{C}{\beta} \int_{\mathbb{R}_{n,+}^m} \|f(y_{1,m}, \cdot)\|_{L_1(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m} = \frac{C}{\beta} \|f\|_{L_{1,\gamma}(\mathbb{R}_{n,+}^N)}. \end{aligned}$$

Consider now the case $1 < p < q < \infty$. Putting $h(y_{1,m}) = \|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})}$ and using Lemma 2, we obtain

$$\begin{aligned} |I_\gamma^\alpha f(x)| & \leq \int_{\mathbb{R}_{n,+}^N} T^y |x|^{\alpha-N-|\gamma|} |f(y)| y_{1,n}^\gamma dy \\ & \leq C_2 \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-|\gamma|-(\frac{N}{p}+\frac{m}{p'})} \|f(y_{1,m}, \cdot)\|_{L_p(\mathbb{R}^{N-m})} y_{1,n}^\gamma dy_{1,m} \\ & = C_2 \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\alpha-|\gamma|-(\frac{N}{p}+\frac{m}{p'})} h(y_{1,m}) y_{1,n}^\gamma dy_{1,m}. \end{aligned}$$

Since $\alpha - |\gamma| - (N/p + m/p') = -(m + |\gamma|)(1/p' + 1/q)$, it follows that

$$\|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)} \leq C_2 \left\| \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma|)(\frac{1}{p'}+\frac{1}{q})} h(y_{1,m}) y_{1,n}^\gamma dy_{1,m} \right\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)}.$$

Since

$$\begin{aligned} & \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma|)(\frac{1}{p'}+\frac{1}{q})} h(y_{1,m}) y_{1,n}^\gamma dy_{1,m} \\ & = \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{\beta-m-|\gamma|} h(y_{1,m}) y_{1,n}^\gamma dy_{1,m} = I_\gamma^\beta h(x_{1,m}), \end{aligned}$$

where $\beta = (m + |\gamma|)(\frac{1}{p} - \frac{1}{q}) > 0$ and $N = m$, applying Theorem 1 for $N = m$ and considering that $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{m+|\gamma|}$, we obtain

$$\begin{aligned} & \|I_\gamma^\alpha f(\cdot, x_{m+1,N})\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)} \left\| \int_{\mathbb{R}_{n,+}^m} T_{x_{1,m}}^{y_{1,m}} |x_{1,m}|^{-(m+|\gamma|)(\frac{1}{p'}+\frac{1}{q})} h(y_{1,m}) y_{1,n}^\gamma dy_{1,m} \right\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)} \\ & = \|I_\gamma^\beta h\|_{L_{q,\gamma}(\mathbb{R}_{n,+}^m)} \leq C \|h\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^m)} = C \|f\|_{L_{p,\gamma}(\mathbb{R}_{n,+}^N)} \end{aligned}$$

with some constant $C > 0$ independent of f and $x_{m+1,N}$.

The proof of Theorem 3 is complete.

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