Georgian Mathematical Journal
Volume 15 (2008), Number 2, 251–262

ON ESTIMATING THE APPROXIMATION OF LOCALLY SUMMABLE FUNCTIONS BY GEGENBAUER SINGULAR INTEGRALS

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Dedicated to Prof. Vakhtang Kokilashvili
on the occasion of his 70th birthday

Abstract. Using the generalized shift operator (GSO) generated by the Gegenbauer differential operator we introduce the notion of a Lebesgue–Gegenbauer (L-G)-point of a summable function f on the interval $[1,\infty)$ and prove that almost all points of this interval are (L-G)-points of f. Furthermore, we give an exact (by order) estimation of the approximation of locally summable functions by singular integrals generated by GSO (Gegenbauer singular integrals).

2000 Mathematics Subject Classification: 42B20, 42B25, 42B35.
Key words and phrases: Gegenbauer differential operator, generalized shift operator, Gegenbauer singular integral.

1. Preliminaries

The generalized shift operator (GSO) generated by the Gegenbauer differential operator

$$D_\lambda = (x^2-1)^{\frac12-\lambda}\left[\frac{d}{dx}(x^2-1)^{\lambda+\frac12}\frac{d}{dx}\right], \quad 0<\lambda<\frac12,$$

which arises from the addition theorem for Gegenbauer functions (see [1]) is of the form

$$A_t f(x) = A_t^\lambda f(x) = \frac{\Gamma\left(\lambda+\frac12\right)}{\Gamma\left(\frac12\right)\Gamma(\lambda)}\int_0^\pi f\left(xt-\sqrt{x^2-1}\sqrt{t^2-1}\cos\varphi\right)(\sin\varphi)^{2\lambda-1}\,d\varphi.$$

We denote by $L_{1,\lambda}[1,\infty)$, $0<\lambda<\frac12$, a class of functions with finite norm

$$\|f\|_{1,\lambda} = \int_1^\infty \left(x^2-1\right)^{\lambda-\frac12}|f(x)|\,dx<\infty.$$

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Definition. Let $f \in L_{1,\lambda}[1,\infty)$, $0<\lambda<\frac{1}{2}$. The point $x \in [1,\infty)$ is said to be a Lebesgue–Gegenbauer point ((L-G)-point) of the function f, if

$$\lim_{r\to 0}\left(\sinh\frac{r}{2}\right)^{-2\lambda-1}\int_0^r \left|f(x)-A^{\lambda}_{\cosh t}f(x)\right|\sinh^{2\lambda} t dt = 0.$$

Theorem 1. *If $f \in L_{1,\lambda}[1,\infty)$, $0<\lambda<\frac{1}{2}$, then almost all points of the interval $[1,\infty)$ are the (L-G)-points of f.*

Proof. Let us estimate the integral

$$J(x,r)=\int_0^r \left|f(x)-A^{\lambda}_{\cosh t}f(x)\right|\sinh^{2\lambda} t dt$$

$$\leq \int_0^r\left[\int_0^\pi \left|f(x)-f\left(x\cosh t-\sqrt{x^2-1}\sinh t\cos\varphi\right)\right|(\sin\varphi)^{2\lambda-1}d\varphi\right]\sinh^{2\lambda}t dt.$$

Setting here $x=\cosh u\in[1,\infty)$, we get

$$J(\cosh u,r)\leq\int_0^r\left[\int_0^\pi |f(\cosh u)-f(\cosh u\ \cosh t-\sinh u\ \sinh\ t\cos\varphi)|\right.$$
$$\left.\times(\sinh\varphi)^{2\lambda-1}d\varphi\right]\sinh^{2\lambda}t dt.$$

Performing the change of variables $z=\cosh u\ \cosh t-\sinh u\sinh\ t\cos\varphi$, we get

$$d\varphi=\left(\sinh^2 u\sinh^2 t-\cosh^2 u\cosh^2 t+2z\cosh\ u\ \cosh t-z^2\right)^{-\frac{1}{2}}dz$$

or

$$d\varphi=\left(2z\cosh u\cosh t-\sinh^2 t-\cosh^2 u-z^2\right)^{-\frac{1}{2}}dz$$

and

$$(\sin\varphi)^{2\lambda-1}=\left(2z\cosh u\ \cosh\ t-\sinh^2 t-\cosh^2 u-z^2\right)^{\lambda-\frac{1}{2}}(\sinh u\sinh t)^{1-2\lambda}.$$

Then the above estimate of $J(\cosh u,r)$ is rewritten as

$$J(\cosh u,r)\leq\int_0^r\left\{\int_{\cosh(u-t)}^{\cosh(u+t)}|f(\cosh u)-f(z)|\right.$$
$$\left.\times\left(2z\cosh u\cdot\cosh t-\sinh^2 t-\cosh^2 u-z^2\right)^{\lambda-1}(\sinh u)^{1-2\lambda}dz\right\}\sinh t dt. \quad (1)$$

It is easy to show that

$$2z\cosh u\ \cosh t-\sinh^2 t-\cosh^2 u-z^2$$

$$= \left(z^2-1\right)\sinh^2 u\left[1-\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right)^2\right]. \quad (2)$$

Having noticed that

$$\frac{\sinh t}{\sinh u} = \left(z^2-1\right)^{1/2}\frac{\partial}{\partial t}\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right) \quad (3)$$

and taking into account (2) and (3) in (1), we get

$$J(\cosh u, r) \le \int\limits_0^r \Bigg\{ \int\limits_{\cosh(u-t)}^{\cosh(u+t)} |f(\cosh u) - f(z)| \left(z^2-1\right)^{\lambda-1}$$

$$\times\left[1-\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right)dz\Bigg\}dt. \quad (4)$$

Since $\cosh(u-t) \le z \le \cosh(u+t)$, we have

$$\begin{cases} \cosh(u-r) \le z \le \cosh u, \\ u - \operatorname{arc}\cosh z \le t \le r, \end{cases} \qquad \begin{cases} \cosh u \le z \le \cosh(u+r), \\ \operatorname{arc}\cosh z - u \le t \le r. \end{cases}$$

Therefore, after changing the integration order in (4), we get

$$J(\cosh u, r) \le \left(\int\limits_{\cosh(u-r)}^{\cosh u} dz \int\limits_{u-arcch\, z}^{r} dt + \int\limits_{\cosh u}^{\cosh(u+r)} dz \int\limits_{\operatorname{arc}\cosh\, z-u}^{r} dt \right)$$

$$= \left(J_1(\cosh u, r) + J_2(\cosh u, r)\right). \quad (5)$$

Let us consider the integral

$$A = \int\limits_{u-arcch\, z}^{r}\left[1-\left(\frac{\cosh t - z\cosh\, u}{\sqrt{z^2-1}\sinh u}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right)dt.$$

Having set here $\vartheta = \frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\ \sinh\, u}$, we get

$$A = \int\limits_{-1}^{\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\ \sinh\, u}} \left(1-\vartheta^2\right)^{\lambda-1} d\vartheta. \quad (6)$$

By the evenness property of $\cosh t$

$$B = \int\limits_{\coth\, z-u}^{r}\left[1-\left(\frac{\cosh t - z\cosh\, u}{\sqrt{z^2-1}\sinh u}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}\right)dt$$

$$= \int\limits_{-1}^{\frac{\cosh r - z\cosh u}{\sqrt{z^2-1}\sinh u}} \left(1-\vartheta^2\right)^{\lambda-1} d\vartheta.$$

Since for $0 < r < u$ and $\cosh(u-r) \le z \le \cosh u$

$$\frac{\cosh r - z\cosh u}{\sqrt{z^2-1}\sinh u} \le \frac{\cosh u - z\cosh u}{\sqrt{z^2-1}} \le \frac{(1-z)\cosh u}{\sqrt{z^2-1}\sinh u} < 0,$$

we have

$$\max_{-1\le\vartheta\le\frac{\cosh t - z\cosh u}{\sqrt{z^2-1}\sinh u}} (1-\vartheta)^{\lambda-1} \le \max_{-1\le\vartheta<0} (1-\vartheta)^{\lambda-1} \le 1.$$

Then (6) implies

$$A \le \frac{1}{\lambda}\left(\frac{\sqrt{z^2-1}\sinh u - z\cosh u - \cosh r}{\sqrt{z^2-1}\sinh u}\right)^{\lambda} \tag{7}$$

and from (7) we get

$$A \le \frac{2^\lambda}{\lambda}\frac{\left(\sinh\frac{r}{2}\right)^{2\lambda}}{(\sinh(u-r)\sinh u)^{\lambda}} \sim \frac{1}{\lambda 2^\lambda}\left(\frac{r}{\sinh u}\right)^{2\lambda} (r\to 0)$$

Hence it follows that

$$A \le C_\lambda\left(\frac{r}{\sinh u}\right)^{2\lambda} \qquad (r\to 0), \tag{8}$$

where C_λ is a constant dependent on λ. In the sequel, C_λ will denote, generally speaking, different constants dependent on λ.

Taking into account (8) in (5), we get that for J_1

$$J_1(\cosh u, r) \le C_\lambda\left(\frac{r}{\sinh u}\right)^{2\lambda}\int_{\cosh(u-r)}^{\cosh u} |f(\cosh u) - f(z)|\left(z^2-1\right)^{\lambda-\frac{1}{2}} dz.$$

Having put $z = \cosh v$, we have

$$J_1(\cosh u, r) \le C_\lambda\left(\frac{r}{\sinh u}\right)^{2\lambda}\int_{u-r}^{u} |f(\cosh u) - f(\cosh v)|\sinh^{2\lambda} v dv.$$

Since

$$\int_1^\infty |f(x)|\left(x^2-1\right)^{\lambda-\frac{1}{2}} dx = \int_0^\infty |f(\cosh u)|\sinh^{2\lambda} u du < \infty,$$

we have

$$f(x)\left(x^2-1\right)^{\lambda-\frac{1}{2}} \in L[1,\infty) \Leftrightarrow f(\cosh u)\sinh^{2\lambda} u \in L[0,\infty).$$

Then almost all the points of the interval $[0,\infty)$ are the ordinary Lebesgue points for the function $\varphi(u) = f(\cosh u)\sinh^{2\lambda} u$ and therefore

$$\int_{u-r}^{u} |\varphi(u) - \varphi(v)| dv = o(r) \qquad (r\to 0),$$

which is equivalent to

$$\int_{u-r}^{u} \left| f(\cosh u) \sinh^{2\lambda} u - f(\cosh v) \sinh^{2\lambda} v \right| dv = o(r) \qquad (r \to 0). \tag{9}$$

Hence it follows that

$$\int_{u-r}^{u} |f(\cosh u) - f(\cosh v)| \sinh^{2\lambda} v dv$$
$$\leq \int_{u-r}^{u} \left| f(\cosh\ u) \sinh^{2\lambda} u - f(\cosh v) \sinh^{2\lambda} v \right| dv$$
$$+ |f(\cosh u)| \left[\sinh^{2\lambda} u - \sinh^{2\lambda} (u-r)\right] r = o(r) \quad (r \to 0). \tag{10}$$

Formulas (9) and (10) imply that

$$J_1(\cosh u, r) = o\left(r^{2\lambda+1}\right) \quad (r \to 0). \tag{11}$$

Let us consider the integral

$$J_2(\cosh u, r)$$
$$= \int_{\cosh u}^{\cosh(u+r)} |f(\cosh u) - f(z)| \left(z^2-1\right)^{\lambda-\frac{1}{2}} dz \int_{-1}^{\frac{\cosh r - z\cosh u}{\sqrt{z^2-1}\sinh u}} (1-v)^{\lambda-1} dv. \tag{12}$$

Since $\cosh u \leq z \leq \cosh(u+r)$, we obtain for $0 < r < u$

$$\frac{\cosh r - z\cosh u}{\sqrt{z^2-1}\sinh u} \leq \frac{\cosh u - z\cosh u}{\sinh^2 u} = \frac{(1-z)\cosh u}{\sinh^2 u} < 0.$$

Then

$$\max_{-1 \leq u \leq \frac{\cosh r - z\cosh u}{\sqrt{z^2-1}\sinh u}} (1-u)^{\lambda-1} \leq \max_{-1\leq u\leq 0} (1-u)^{\lambda-1} \leq 1,$$

and therefore from (12) we have

$$B \leq \frac{1}{\lambda}\left(\frac{2\sinh^2 \frac{r}{2}}{\sinh^2 u}\right)^{\lambda} \sim \frac{1}{\lambda 2^{\lambda}}\left(\frac{r}{\sinh u}\right)^{2\lambda} \quad (r \to 0). \tag{13}$$

Taking into account (13) in (12) and having put $z = \cosh v$, we get

$$J_2(\cosh u, r) \leq C_\lambda \left(\frac{r}{\sinh u}\right)^{2\lambda} \int_{\cosh u}^{\cosh(u+r)} |f(\cosh u) - f(\cosh v)| \sinh^{2\lambda} v dv. \tag{14}$$

Proceeding as above, we get

$$\int_{\cosh u}^{\cosh(u+r)} |f(\cosh\ u) - f(\cosh v)| \sinh^{2\lambda} v dv$$

$$\leq o(r) + |f(\cosh u)| \left(\sinh^{2\lambda}(u+r) - \sinh^{2\lambda} u\right) r = o(r) \quad (r \to 0). \quad (15)$$

From (14) and (15) it follows that

$$J_2(\cosh u, r) = o\left(r^{2\lambda+1}\right) \quad r \to 0. \quad (16)$$

Finally, taking into account (15) and (16) in (7), we get

$$J(\cosh u, r) = o\left(r^{2\lambda+1}\right) \quad r \to 0,$$

which is equivalent to the assertion of the theorem since $\sin r \sim r$ $(r \to 0)$. □

2. The Main Results

$L_{1,\lambda}[1,\infty)$ is the class of functions summable with respect to the measure $d\mu(u) = (u^2-1)^{\lambda-\frac{1}{2}} du$ in the Lebesgue sense on $[1,\infty)$. We denote by $L_{loc,\lambda}[1,\infty)$ a class of locally summable functions with respect to the measure $d\mu(u)$. A function $K_\varepsilon^\lambda(x)$ determined on $[1,\infty)$ and dependent on the parameter ε is called a kernel, if $K_\varepsilon^\lambda(x) \in L_{1,\lambda}[1,\infty)$ and the equality

$$\int_1^\infty (u^2-1)^{\lambda-\frac{1}{2}} K_\varepsilon^\lambda(u) du = 1 \quad (17)$$

is fulfilled for any fixed ε. If $K_\varepsilon^\lambda(u)$ is a kernel, then the expression

$$L_\varepsilon^\lambda f(x) = \int_1^\infty (u^2-1)^{\lambda-\frac{1}{2}} A_u^\lambda f(x) K_\varepsilon^\lambda(u) du \quad (18)$$

is said to be a Gegenbauer singular integral.

Denote $K_\varepsilon^*(x) = \sup\limits_{y \geq x} \left|K_\varepsilon^\lambda(y)\right|$. Clearly, $K_\varepsilon^*(x)$ monotonically decreases on $[1,\infty)$ with respect to x.

Let $x_0 \in [1,\infty)$ and $f \in L_{loc,\lambda}[1,\infty)$. Put

$$w_f(x_0;\delta)_\lambda = \sup_{0<r<\delta} \left(\frac{1}{\left(\sinh\frac{r}{2}\right)^{2\lambda+1}} \int_0^r \left|A_{\cosh u}^\lambda f(x_0) - f(x_0)\right| \sinh^{2\lambda} u du \right).$$

It follows from Theorem 1 that $\lim\limits_{\delta\to 0} w_f(x_0;\delta)_\lambda = 0$ for all $x \in [1,\infty)$.

We denote by Ω the class of positive, monotonically increasing functions $\omega(t)$ on $[1,\infty)$ such that $\lim\limits_{t\to 0} \omega(t) = 0$

$$L_{x_0,w,\lambda}[1,\infty) = \left\{f \in L_\lambda[1,\infty) : w_f(x_0;\delta)_\lambda = O(w(\delta)), \delta > 0\right\}, w \in \Omega.$$

The norm in this class is defined as follows:

$$\|f\|_{L_{x_0,w,\lambda}} = |f(x_0)| + \sup_{\delta>0} \frac{w_f(x_0;\delta)_\lambda}{w(\delta)}.$$

Theorem 2. *Let* $x_0 \in [1, \infty)$, $f \in L_\lambda[1, \infty)$. *Then the estimate*

$$\left|L_\varepsilon^\lambda f(x_0) - f(x_0)\right| \leq 4^{2\lambda+1} \int_0^\infty K_\varepsilon^*(\cosh u) w_f(x_0; 4u)_\lambda \sinh^{2\lambda} u du \quad (\varepsilon > 0) \quad (19)$$

is valid.

Proof. From (17) and (18) we have

$$\left|L_\varepsilon^\lambda f(x_0) - f(x_0)\right| \leq \sum_{k=0}^\infty K_\varepsilon^*(\cosh 2^k\varepsilon)(2^{k+1}\varepsilon)^{2\lambda+1} w_f(x_0; 2^{k+1}\varepsilon)_\lambda. \quad (20)$$

On the other hand,

$$\int_0^\infty K_\varepsilon^*(\cosh u)\, w_f(x_0; 4u)_\lambda \sinh^{2\lambda} u du$$
$$\geq 4^{-2\lambda-1} \sum_{k=0}^\infty K_\varepsilon^*\left(\cosh 2^k\varepsilon\right) w_f\left(x_0; 2^{k+1}\varepsilon\right)_\lambda \left(2^{k+1}\varepsilon\right)^{2\lambda+1}. \quad (21)$$

From (20) and (21) we obtain (19). □

Corollary 1. *Let* $w, w_1 \in \Omega$ *and*

$$\int_0^\infty K_\varepsilon^*(\cosh u) w(4u) \sinh^{2\lambda} u du = O(w_1(\varepsilon)) \quad (\varepsilon > 0).$$

Then for $f \in L_{x_0,w,\lambda}[1, \infty)$ *the estimate*

$$\left|L_\varepsilon^\lambda f(x_0) - f(x_0)\right| \leq C \|f\|_{L_{x_0,w,\lambda}} w_1(\varepsilon)$$

holds, where $C > 0$ *is a constant independent of* f *and* ε.

Indeed,

$$\int_0^\infty K_\varepsilon^*(\cosh u) w_f(x_0; 4u)_\lambda \sinh^{2\lambda} u du$$
$$\leq \|f\|_{L_{x_0,w,\lambda}} \int_0^\infty K_e^*(\cosh u)\, w\,(4u) \sinh^{2\lambda} u du \leq C \|f\|_{L_{x_0,w,\lambda}} w_1(\varepsilon).$$

Let us consider the Gauss–Weierstrass kernel type function

$$W_\varepsilon^\lambda(\cosh t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)\varepsilon} e^{-\left(\frac{\sinh t}{2\varepsilon}\right)^2} \frac{\cosh t}{\sinh^{2\lambda} t}.$$

We are to show that this function satisfies condition (17), i.e. it is a kernel. Indeed, having put $K_\varepsilon^\lambda(\cosh\, t) = W_\varepsilon^\lambda(\cosh t)$, we get

$$\int_0^\infty W_\varepsilon^\lambda(\cosh t)\sinh^{2\lambda} t dt = \frac{1}{\Gamma\left(\frac{1}{2}\right)\varepsilon}\int_0^\infty e^{-\left(\frac{\sinh\, t}{2\varepsilon}\right)^2}\cosh t dt = 1.$$

Corollary 2. *For the kernel $W_\varepsilon^\lambda(\cosh t)$ there holes the inequality*

$$\int_0^\infty W_\varepsilon^\lambda(\cosh t)\, w_f(x_0, 4t)_\lambda \sinh^{2\nu} t dt \le \frac{65\varepsilon}{4\sqrt{\pi}}\int_0^\infty \frac{w_f(x_0, t)_\lambda \sinh^{2\lambda} t dt}{t^{2\lambda+2}}. \tag{22}$$

Indeed,

$$\int_0^\infty W_\varepsilon^\lambda(\cosh t)\, w_f(x_0, 4t)_\lambda \sinh^{2\lambda} t dt$$

$$\le \frac{1}{\Gamma\left(\frac{1}{2}\right)\varepsilon}\int_0^\infty e^{-\left(\frac{\sinh t}{2\varepsilon}\right)^2} w_f(x_0; 4\sinh\, t)_\lambda\, d\sinh t$$

$$\le \frac{1}{4\sqrt{\pi}}\left(\frac{1}{\varepsilon}\int_0^\varepsilon w_f(x_0; t)_\lambda\, dt + 64\varepsilon\int_\varepsilon^\infty \frac{w_f(x_0; t)_\lambda\, dt}{t^2}\right). \tag{23}$$

Furthermore,

$$\frac{1}{\varepsilon}\int_0^\varepsilon w_f(x_0; t)_\lambda\, dt \le \varepsilon\int_\varepsilon^\infty \frac{w_f(x_0; x)_\lambda}{x^2}dx. \tag{24}$$

Taking into account (23) in (22), we get

$$\int_0^\infty W_\varepsilon^\lambda(\cosh t)\, w_f(x_0; 4t)_\lambda \sinh^{2\lambda} t dt \le \frac{65}{4\sqrt{\pi}}\,\varepsilon\int_\varepsilon^\infty \frac{w_f(x_0; t)_\lambda \sinh^{2\lambda} t dt}{t^{2+2\lambda}}.$$

Note that an inequality of type (22) was proved for Poisson kernel in [3] in the periodic case, and in [2] in the non-periodic case.

If in integral (18) we put $K_\varepsilon^\lambda(\cosh u) = \frac{1}{\varepsilon}\varphi(\cosh\frac{u}{\varepsilon})$, then we get $K_\varepsilon^*(\cosh\, u) = \frac{1}{\varepsilon}\varphi^*\left(\cosh\frac{u}{\varepsilon}\right)$, where $\varphi^*(\cosh x) = \sup\limits_{u\ge x}|\varphi(\cosh\, u)|$.

Theorem 3. *Let $f \in L_\lambda[1,\infty)$, $x_0 \in [1,\infty)$,*

$$K_\varepsilon^\lambda(\cosh u) = \frac{1}{\varepsilon}\varphi\left(\cosh\frac{u}{\varepsilon}\right),\ \int_0^\infty \varphi^*(\cosh\, u)\sinh^{2\lambda} u du < \infty$$

and

$$\int_0^\infty \varphi^*(\cosh u) w_f(x_0; 4u)_\lambda \sinh^{2\lambda} u du < \infty.$$

Then

$$\lim_{\varepsilon\to 0} L_\varepsilon^\lambda f(x_0) = f(x_0).$$

Proof. Let $0<\varepsilon\le 1$ $K_\varepsilon^\lambda(\cosh u)=\frac{1}{\varepsilon}\varphi(\cosh\frac{u}{\varepsilon})$. Then we have

$$\int_0^\infty K_\varepsilon^*(\cosh u) w_f(x_0;4u)_\lambda \sinh^{2\lambda} u du \le w_f(x_0;4\sqrt{\varepsilon})_\lambda \int_0^{\frac{1}{\sqrt{\varepsilon}}} \varphi^*(\cosh t)\sinh^{2\lambda} t dt$$
$$+\int_{\frac{1}{\sqrt{\varepsilon}}}^\infty \varphi^*(\cosh t) w_f(x_0;4t)_\lambda \sinh^{2\lambda} t dt. \quad (25)$$

The assertion of the theorem follows from (19) and (25). □

Note that the operator A_t^λ is bounded in the class of functions bounded on $[1,\infty)$, i.e., $\left|A_t^\lambda f(x)\right| \le \sup_{x\ge 1}|f(x)|$.

Put $w_k(f,t)=\sup\limits_{x\ge 1}\left|A_t^k f(x)-f(x)\right|$, $k=1,2,\ldots$, $w(f,t)\equiv w_1(f,t)$, $A_t^k = A_t\left(A_t^{k-1}\right)$.

Hence we get

$$w_2(f,t)\le 2\sup_{x\ge 1}\left|A_t^\lambda f(x)-f(x)\right| = 2w(f,t). \quad (26)$$

Theorem 4. *Let $x_0\in[1,\infty)$, the kernel $K_\varepsilon^\lambda(t)$ non-negatively and monotonically decrease on $[1,\infty)$ with respect to t.*

Then there exists a function $f_0\in L_{x_0,w,\lambda}[1,\infty)$ such that

$$w_{f_0}(x_0;\delta)_\lambda \asymp w(\delta) \quad (\delta\to 0)^1$$

and

$$\left|L_\varepsilon^\lambda f_0(x_0)-f(x_0)\right| \ge (2\lambda+1)\int_0^\infty K_\varepsilon^\lambda(\cosh\, t) w_{f_0}(x_0;4t)_\lambda \sinh^{2\lambda} t dt.$$

Proof. Let us consider the function

$$f_0(t)=\begin{cases} A_t^\lambda f(x_0)-f(x_0)=w(t), & t>0,\\ 0, \ t=0.\end{cases}$$

Then

$$w_{f_0}(x_0;\delta)_\lambda = \sup_{0<r<\delta}\left(\frac{1}{\left(\sinh\frac{r}{2}\right)^{2\lambda+1}}\int_0^r w(t)\sinh^{2\lambda} t dt\right). \quad (27)$$

Now, let us estimate the integral

$$J(r)=\int_0^r \sinh^{2\lambda} t dt \ge \frac{2^{\lambda+3/2}}{(2\lambda+1)\left(1+\cosh 1\right)^{\frac{1}{2}-\lambda}}\left(\sinh\frac{r}{2}\right)^{2\lambda+1}. \quad (28)$$

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[1]The symbol $\varphi(\delta)\asymp\psi(\delta)$ $(\delta\to 0)$ means that there exist constans c_1, c_2 and δ_0 such that $c_1\varphi(\delta)\le\psi(\delta)\le c_2\varphi(\delta)$ for $0<\delta<\delta_0$.

On the other hand,

$$\int_0^r \sinh^{2\lambda} t dt \leq \frac{2^{2\lambda+1}}{2\lambda+1}\left(\sinh\frac{r}{2}\right)^{2\lambda+1}. \tag{29}$$

Since $\omega(f,t)$ monotonically increases, (29) implies that

$$\omega_{f_0}(x_0;\delta)_\lambda \leq \frac{2^{2\lambda+1}}{2\lambda+1}\sup_{0<r\leq\delta}\omega(f,r) = \frac{2^{2\lambda+1}}{2\lambda+1}\omega(f,\delta). \tag{30}$$

From (27) we have

$$\omega_{f_0}(x_0;\delta)_\lambda \geq \frac{1}{2}\sup_{0<r\leq\delta}\frac{\omega(f,r)}{\left(\sinh\frac{r}{2}\right)^{2\lambda+1}}\left(\int_0^r - \int_0^{\frac{r}{2}}\right)\sinh^{2\lambda} t dt.$$

Taking into account here (28) and (29), we get

$$\omega_{f_0}(x_0;\delta)_\lambda \geq \frac{2^{2\lambda}}{2\lambda+1}\left(1-\left(\cosh\frac{1}{2}\right)^{2\lambda-1}\right)\omega(f,\delta). \tag{31}$$

From (29) and (31) we have

$$\frac{2^{2\lambda}}{2\lambda+1}\left(1-\left(\cosh\frac{1}{2}\right)^{2\lambda-1}\right)\omega(f,\delta) \leq \omega_{f_0}(x_0;\delta)_\lambda \leq \frac{2^{2\lambda+1}}{2\lambda+1}\omega(f,\delta).$$

Then

$$\left|L_\varepsilon^\lambda f_0(x_0) - f(x_0)\right| \geq \frac{(2\lambda+1)}{2^{2\lambda+1}}\int_0^\infty K_\varepsilon^\lambda(\cosh t) w_{f_0}(x_0;4t)_\lambda \sinh^{2\lambda} t dt. \qquad \square$$

This theorem shows that estimate (19) in Theorem 2 is final with respect to order.

Assume

$$\hat{f}(\alpha) = \int_1^\infty f(x)P_\alpha^\lambda(x)(x^2-1)^{\lambda-\frac{1}{2}}dx, \quad \check{g}(x) = \int_1^\infty g(\alpha)P_\alpha^\lambda(x)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha.$$

Theorem 5. *For any $f, g \in L_{1,\lambda}[1,\infty)$ the equality*

$$\int_1^\infty \hat{f}(\alpha)g(\alpha)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha = \int_1^\infty f(x)\check{g}(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$

is true.

Proof. Applying Fubini's theorem, we get

$$\int_1^\infty \hat{f}(\alpha)g(\alpha)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$=\int_1^\infty \left(\int_1^\infty f(x)P_\alpha^\lambda(x)(x^2-1)^{\lambda-\frac{1}{2}}\right)g(\alpha)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$=\int_1^\infty \left(\int_1^\infty g(\alpha)P_\alpha^\lambda(x)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha\right)f(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$
$$=\int_1^\infty \check{g}(x)f(x)(x^2-1)^{\lambda-\frac{1}{2}}dx. \quad \square$$

Let $\Phi \in L_{1,\lambda}[1,\infty)$. Consider the Φ-means of the Gegenbauer integrals (18)

$$S_\varphi(x,\varepsilon;f)=\int_1^\infty \hat{f}(\alpha)Q_\alpha^\lambda(x)\Phi(\varepsilon\alpha)(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha.$$

Theorem 6. *Let f,Φ and $\varphi=\check{\Phi}$ belong to $L_{1,\lambda}[1,\infty)$,*

$$\int_1^\infty \varphi(t)(t^2-1)^{\lambda-\frac{1}{2}}dt=1,\ x_0\in[1,\infty).$$

Then

$$|S_\varphi(x_0,\varepsilon;f)-f(x_0)|\le 4^{2\lambda+1}\int_0^\infty \varphi^*(\cosh t)w_f(x_0;4\varepsilon t)\sinh^{2\lambda}t dt \qquad (\varepsilon>0)$$

Proof. Denoting $d\mu(t)=(t^2-1)^{\lambda-1/2}dt$ and using Theorem 4 we get

$$S_\varphi(x_0,\varepsilon;f)=\int_1^\infty A_t^\lambda f(x)\varphi_\varepsilon(x)d\mu(x)=\int_1^\infty f(x)A_t^\lambda\varphi_\varepsilon(x)d\mu(x),$$

whence, allowing for the equality

$$\int_1^\infty \varphi_\varepsilon(t)(t^2-1)^{\lambda-\frac{1}{2}}dt=\int_0^\infty \varphi_\varepsilon(\cosh t)\sinh^{2\lambda}t dt=1,$$

we get

$$S_\varphi(x_0,\varepsilon;f)-f(x_0)=\int_0^\infty \left[A_{\cosh t}^\lambda f(x_0)-f(x_0)\right]\varphi_\varepsilon(\cosh t)\sinh^{2\lambda}t dt.$$

Hence, as in the proof of Theorem 2, we get

$$|S_\varphi(x_0,\varepsilon,f)-f(x_0)|\le 4^{2\lambda+1}\int_0^\infty \varphi_\varepsilon^*(\cosh\, u)w_f(x_0;4u)_\lambda \sinh^{2\lambda} u du.$$

Taking into account that here

$$\varphi_\varepsilon(\cosh t)=\frac{\frac{1}{\varepsilon}\varphi\left(\cosh\frac{t}{\varepsilon}\right)\sinh^{2\lambda}\frac{t}{\varepsilon}}{\sinh^{2\lambda} t},$$

we get

$$|S_\varphi(x_0,\varepsilon;f)-f(x_0)|\le 4^{2\lambda+1}\int_0^\infty w_f(x_0,4\varepsilon t)_\lambda \varphi^*(\cosh t)\sinh^{2\lambda} t dt. \qquad \square$$

Remark. Note that the results obtained in Theorem 6 are the analogues of the corresponding results from [2] and in the proof we have used the techniques of the authors of this work.

Acknowledgement

The first author was research partially supported by the grant of the Azerbaijan-U. S. Bilateral Grants Program II (project ANSF Award / AZM1-3110-BA-08), by the grant of Tubitak (project B.02.TBT.0.06.01-220.01-619-48891) and by the grant of INTAS (project 05-1000008-8157).

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(Received 26.09.2007)

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